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Hamiltonian and Non-Hamiltonian Reductions of Charged Particle Dynamics: Diffusion and Self-Organization

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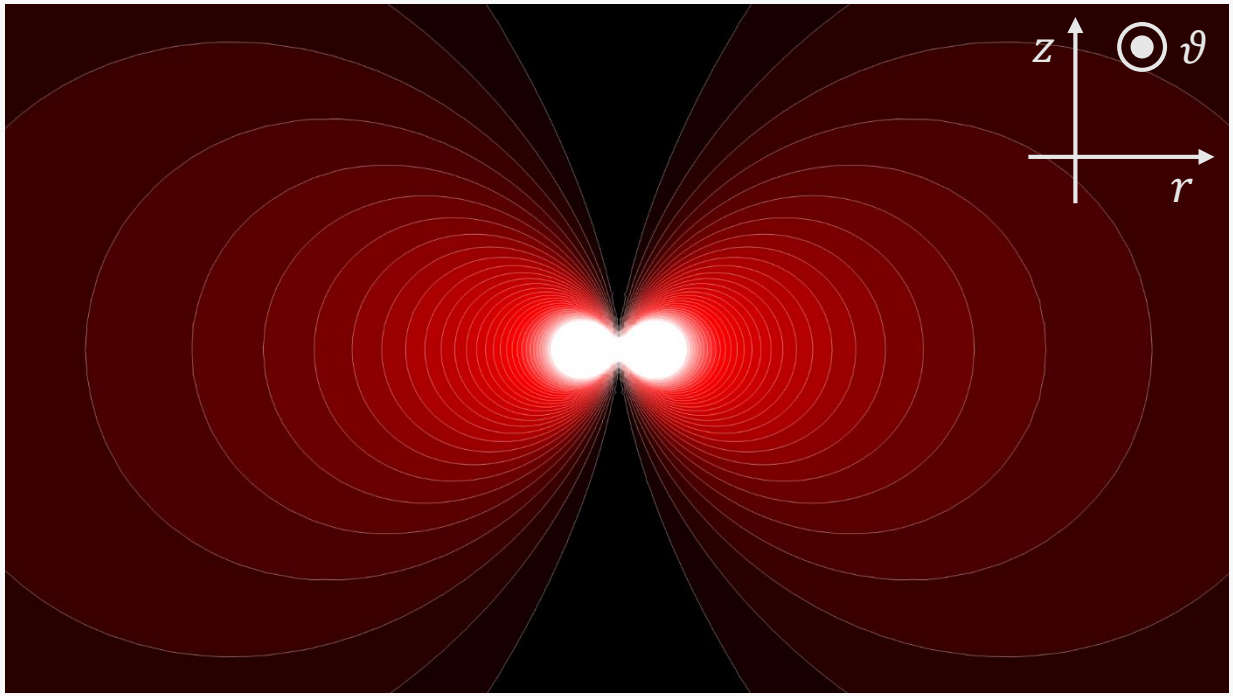


- Magnetospheres are ubiquitous in the universe.
- Dipole magnetic fields confine charged particles.
- Spontaneous confinement (**self-organization**) is achieved by **inward diffusion**.

MAGNETOSPHERIC CONFINEMENT OF CHARGED PARTICLES

- Magnetospheric confinement is a promising source of applications.
- Advanced fusion reactor based on D- ^3H reaction [1].
- Trap for matter and antimatter plasmas [2].
- Study of astrophysical systems [3].

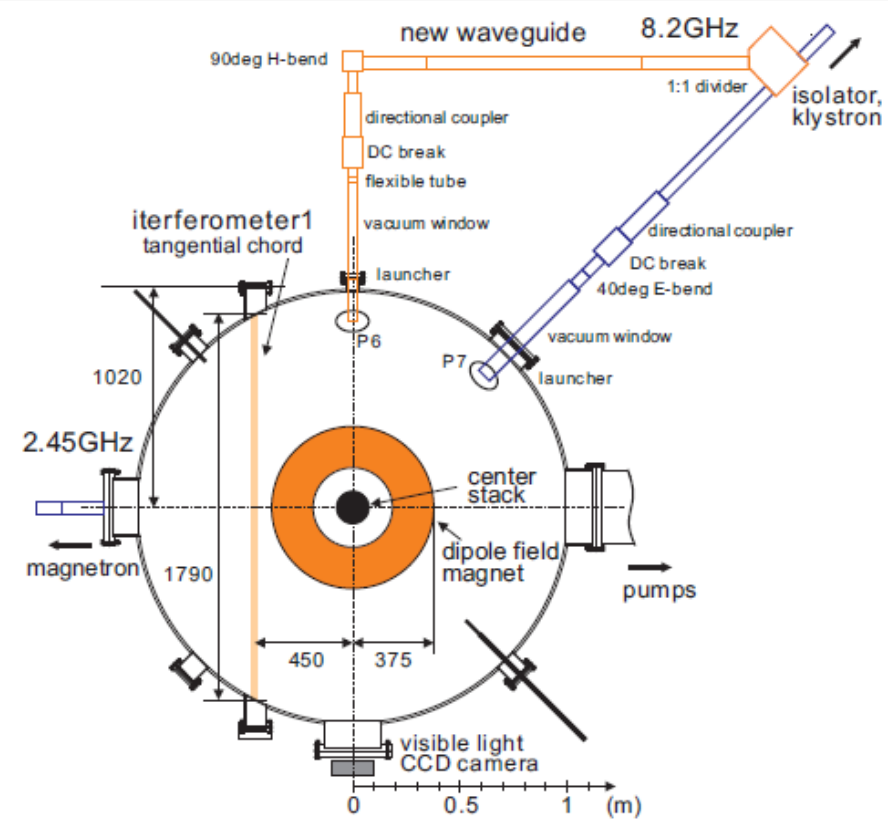
- The creation of the density gradient seemingly violates the entropy principle.

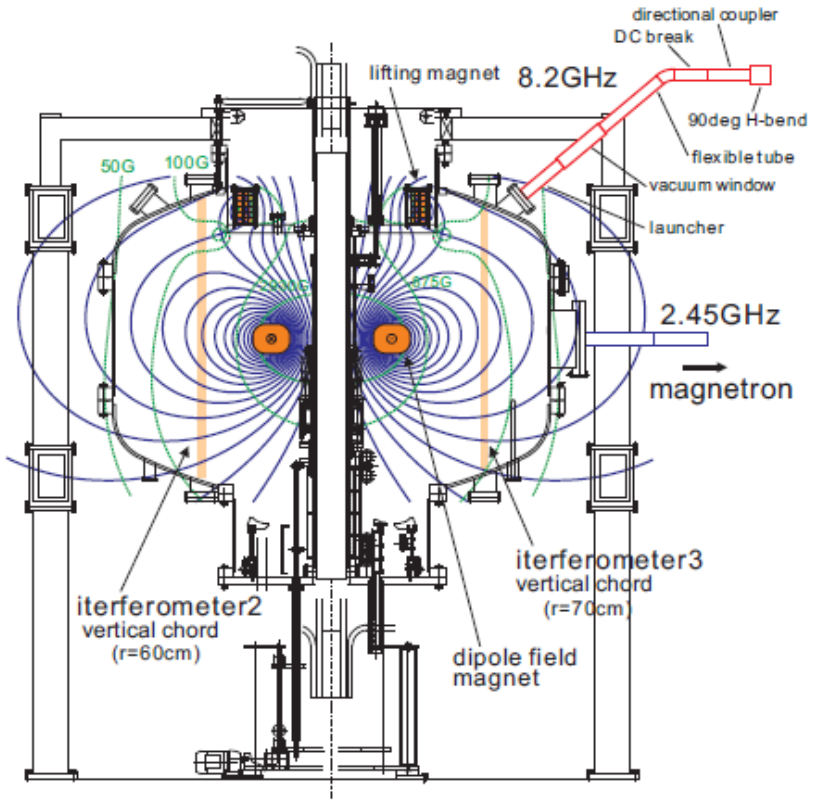
OUTLINE

1. Diffusion in a dipole field: correct expression of diffusion operator in magnetic coordinates.
2. Diffusion in non-integrable magnetic field $\boldsymbol{B} \cdot \nabla \times \boldsymbol{B} \neq 0$.

[1] A. Hasegawa et al., *A D- ^3H Fusion Reactor Based on a Dipole Magnetic Field*, Nuc. Fus. **30**, 11 (1990).
[2] T. S. Pedersen and A. H. Boozer, *Confinement of Nonneutral Plasmas on Magnetic Surfaces*, Phys. Rev. Lett. **88**, 20 (2002).
[3] A. M. Persoon et al., *The Plasma Density Distribution in the Inner Region of Saturn's Magnetosphere*, J. Geo. Res.: Space Physics **118**, 2970-2974 (2013).

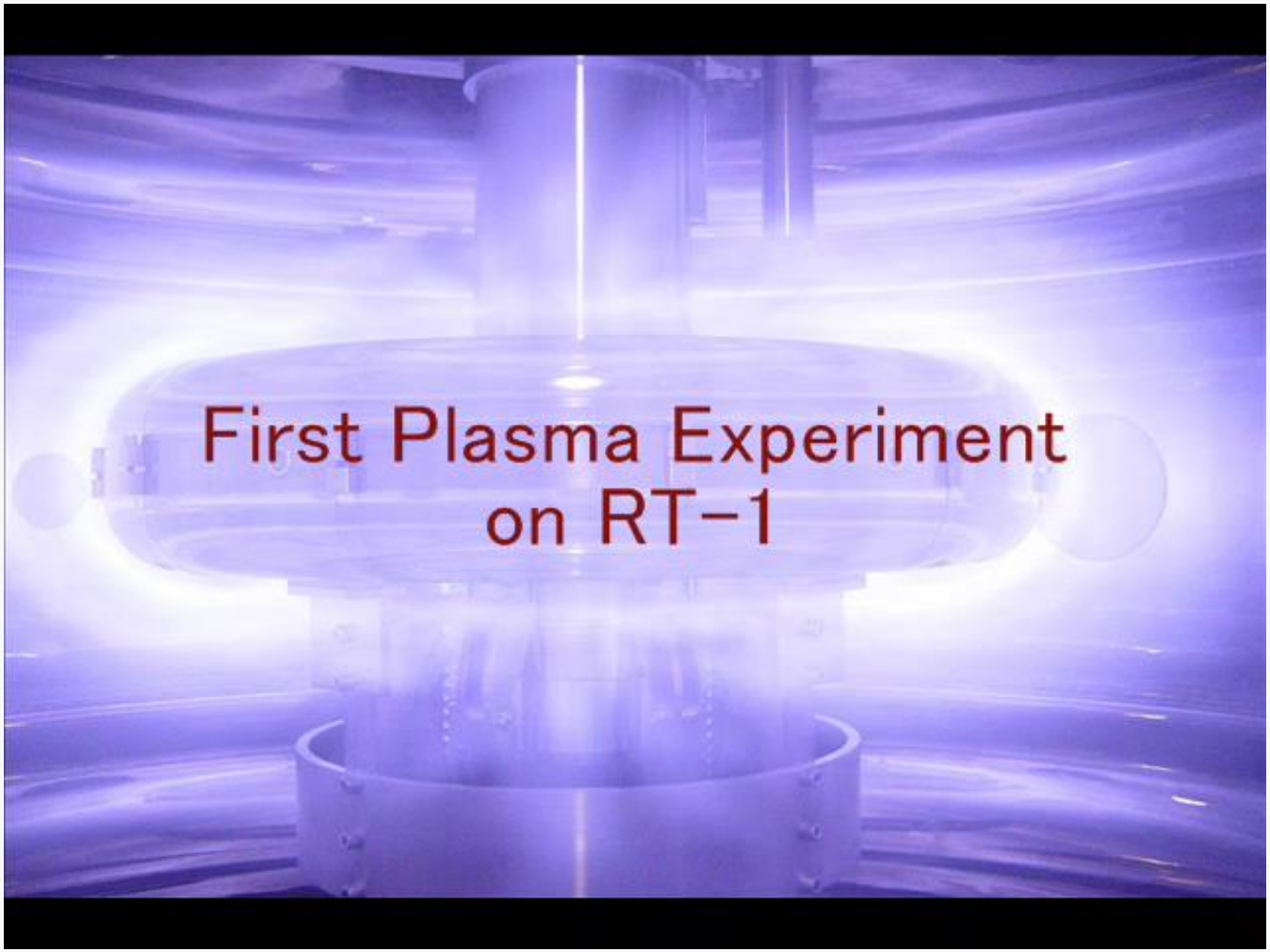
THE RT-1 MACHINE (I)



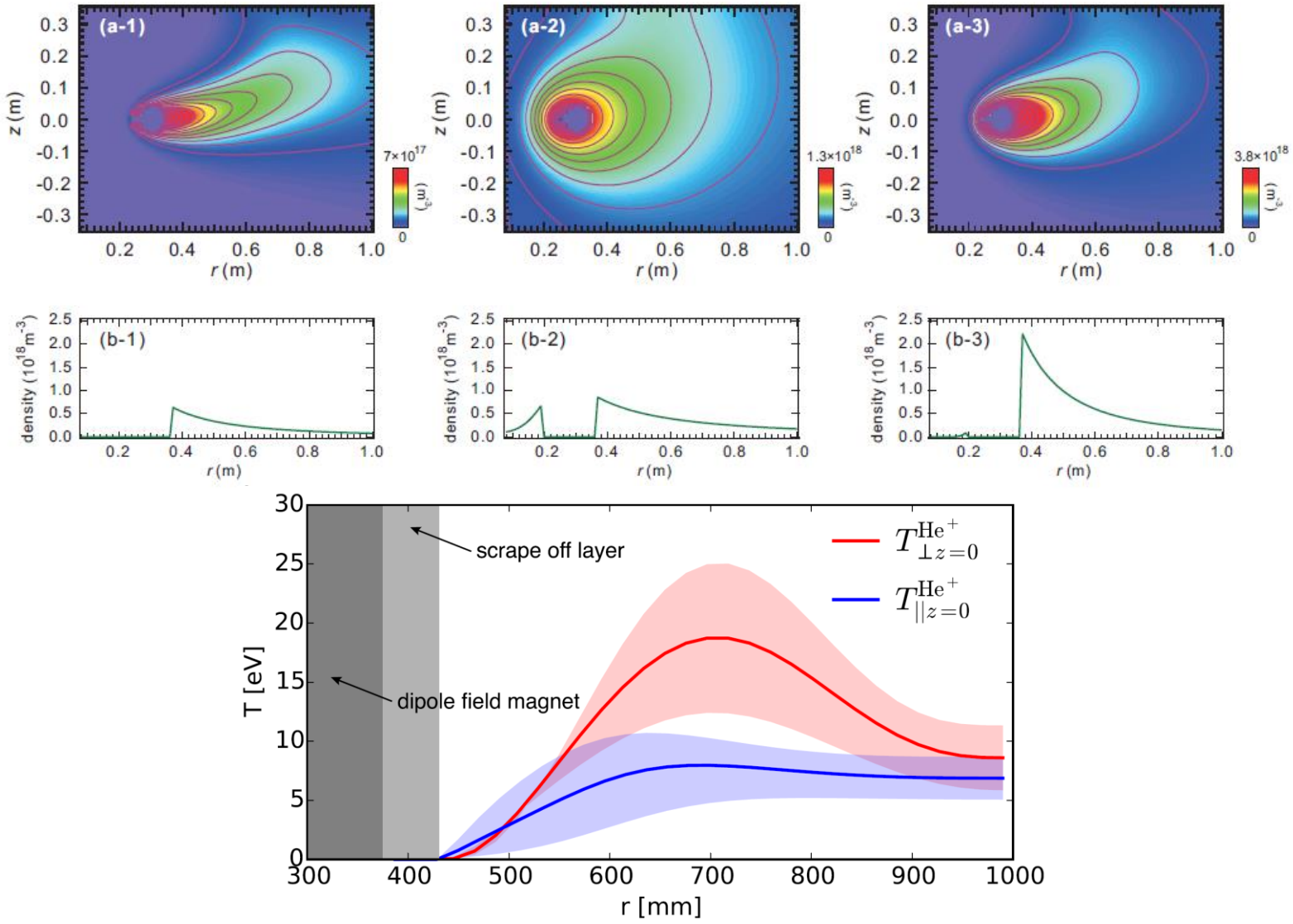


- RT-1 produces a laboratory magnetosphere.
- A similar device is LDX at MIT.

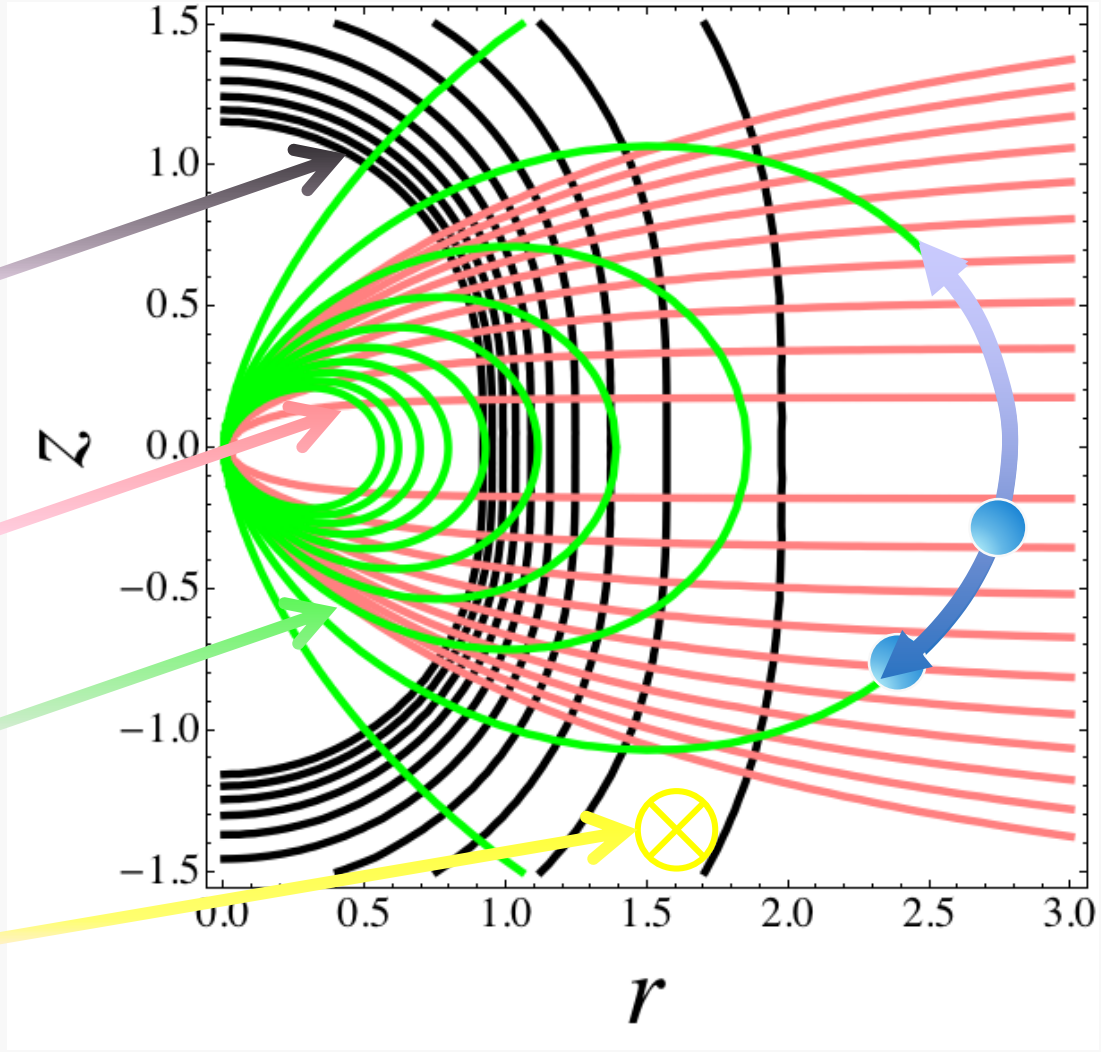
[4] Z. Yoshida et al., *Self-organized confinement by magnetic dipole: recent results from RT-1 and theoretical modeling*, Plasma Phys. Control. Fus. **55** 1 (2012).
[5] Z. Yoshida et al., *Magnetospheric Vortex Formation: Self-Organized Confinement of Charged Particles*, Phys. Rev. Lett. **104**, 235004 (2010).


First Plasma Experiment
on RT-1

EXPERIMENTAL OBSERVATIONS



[6] H. Saitoh et al., *Observation of a New High-β and High-Density State of a Magnetospheric Plasma in RT-1*, Phys. Plasmas **21**, 082411 1-6 (2014).
[7] Y. Kawazura et al., *Observation of Particle Acceleration in Laboratory Magnetosphere*, Phys. Plasmas **22**, 112503 (2015).



B

l

ψ

ϑ

MAGNETIC FIELD

$$\boldsymbol{B} = \nabla\psi \times \nabla\vartheta$$

FLUX FUNCTION

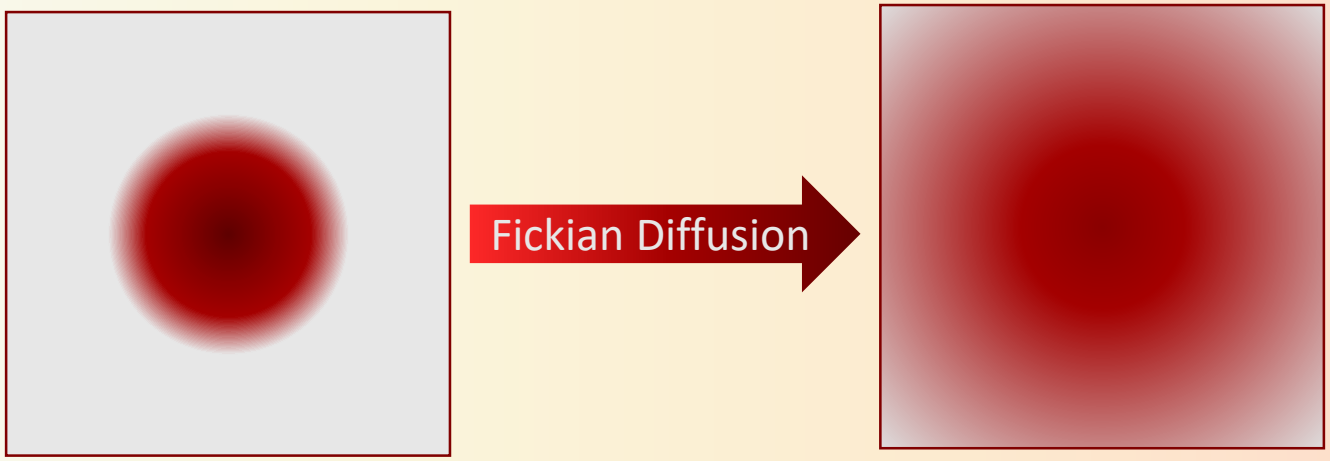
$$\psi(r,z) = \frac{r^2}{(r^2+z^2)^{3/2}}$$

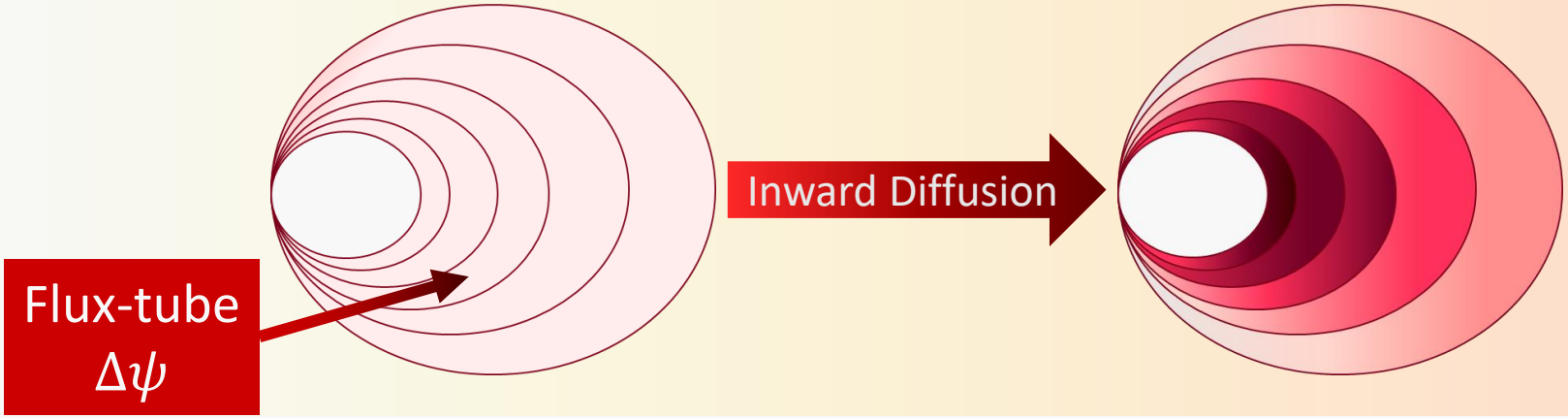
LENGTH OF A FIELD LINE [7]

$$l(r,z) = \frac{1}{2\psi}\left[\frac{1}{\sqrt{3}}\log\left(\sqrt{3}\sqrt{1-(r\psi)^{2/3}}\right.\right.$$

[8] N. Sato and Z. Yoshida, *A Stochastic Model of Inward Diffusion in Magnetospheric Plasmas*, J. Phys. A: Math. Theor **48** 205501 (2015).

INWARD DIFFUSION





It is thought that particle number per flux-tube volume tends to be homogenized [9,10].

[9] A. C. Boxer et al., *Turbulent Inward Pinch of Plasma Confined by a Levitated Dipole Magnet*, Nature Physics, Vol. 6, 207-212 (2010).
[10] A. Hasegawa, *Motion of a Charged Particle and Plasma Equilibrium in a Dipole Magnetic Field …*, Phys. Scr. T 116 72-74 (2005).

- In a dipole field the motion of a charged particle is constrained by 3 adiabatic invariants:

$$\psi \qquad J_{\parallel} \qquad \mu$$

STRENGTH OF CONSTRAINT

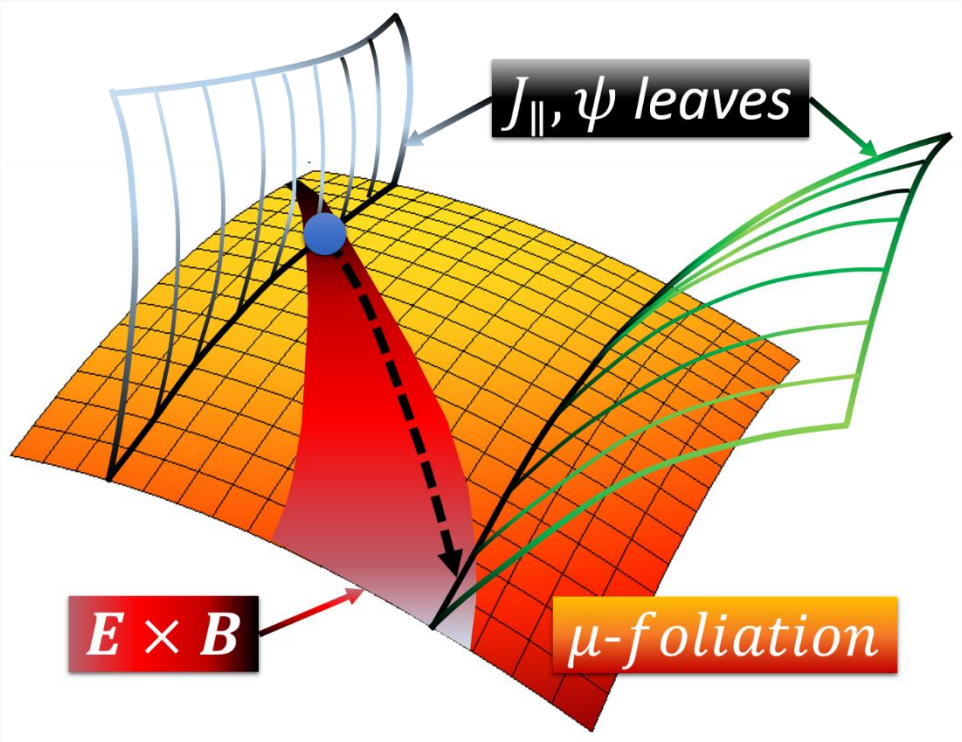
- Each invariant is preserved on a different time scale:

$$\omega_d \ll \omega_b \ll \omega_c$$

- In a dipole plasma the key normal transport mechanism is diffusion by $\boldsymbol{E} \times \boldsymbol{B}$ drift:

$$v_{\perp} = \frac{E_{\perp}}{B}.$$

DIFFUSION ON A FOLIATED PHASE SPACE



- Conservation of adiabatic invariants **foliates phase space**: particle motion is constrained on leaves [11,12].

- Electromagnetic **fluctuations break the weakest topological constraints**. The result is a random process among leaves.

- A diffusion operator must be formulated on the corresponding invariant measure [8,13].
- Entropy should be maximized on the symplectic leaf [14].

[8] N. Sato and Z. Yoshida, *A Stochastic Model of Inward Diffusion in Magnetospheric Plasmas*, J. Phys. A: Math. Theor **48** 205501 (2015).
[11] N. Sato, N. Kasaoka, and Z. Yoshida, *Thermal Equilibrium of Non-Neutral Plasma in Dipole Magnetic Field*, Phys. Plasmas **22** (2015), 042508.
[12] Z. Yoshida and S. M. Mahajan, *Self-Organization in Foliated Phase Space: Construction of a Scale Hierarchy by …*, Prog. Theor. Exp. Phys. **2014** (2014), 073J01
[13] N. Sato, Z. Yoshida, and Y. Kawazura, *Self Organization and Heating by Inward Diffusion in Magnetospheric Plasmas*, Plasma Fus. Res. **11** 2401009 (2016).
[14] N. Sato and Z. Yoshida, *Up-Hill Diffusion, Creation of Density Gradients: Entropy Measure for Systems with Topological* Constraints, Phys. Rev. E **93** 062140 (2016).

- The constraint causing non-canonicality is the conservation of μ: the pair (ϑ_c, μ) can be **reduced**:

$$dx\, dy\, dz\, dp_x\, dp_y\, dp_z = m^2 dl\, dv_\parallel\, d\vartheta\, d\psi\, [d\vartheta_c\, d\mu\,].$$

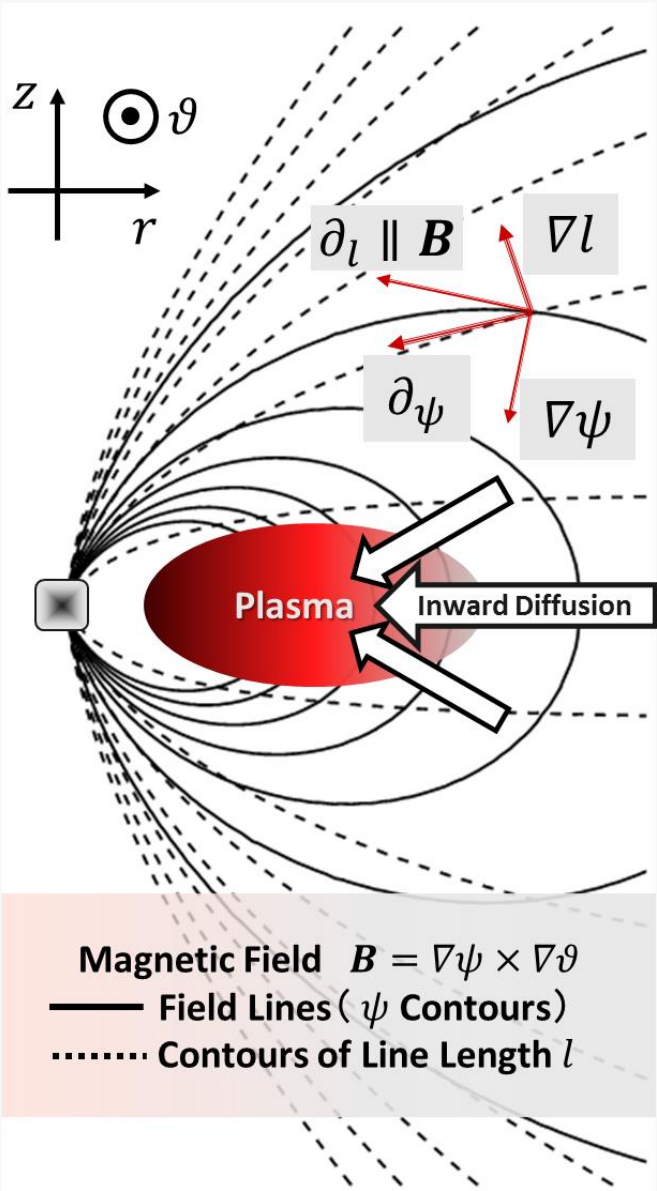
- Dynamical variables are reduced to four: $(l, v_\parallel, \vartheta, \psi)$.

- The reduced equations are the guiding center equations:

$$\begin{cases} \dot{v}_\parallel = -m^{-1}(\mu B + e\varphi)_l + v_\parallel \boldsymbol{v}_{\boldsymbol{E}\times\boldsymbol{B}} \cdot \boldsymbol{k}, \\ \boldsymbol{v} = \boldsymbol{v}_\parallel + \boldsymbol{v}_{\boldsymbol{E}\times\boldsymbol{B}} + \boldsymbol{v}_{\nabla \mathrm{B}} + \boldsymbol{v}_{\boldsymbol{k}}. \end{cases}$$

- Transforming to the new coordinates:

$$\begin{cases} \dot{l} = v_\parallel - q\varphi_\vartheta, \\ \dot{v}_\parallel = -m^{-1}(\mu B + e\varphi)_l + v_\parallel q_l \varphi_\vartheta, \\ \dot{\vartheta} = \left(\partial_\psi + q\partial_l\right)(e^{-1}\mu B + \varphi) - e^{-1} m v_\parallel^2 q_l, \\ \dot{\psi} = -\varphi_\vartheta. \end{cases}$$



- The reduced system should be Hamiltonian because the reduced variable μ is a dynamical constant and the phase ϑ_c can be neglected.

- We look for the nondegenerate closed two-form $\omega = d\lambda$ such that:

$$i_X \omega = -dH_0,$$

with $H_0 = \mu B + \frac{m}{2} v_\parallel^2 + e\varphi$ the Hamiltonian function. Setting $q = -\partial_l \cdot \partial_\psi$:

$$\omega = m dv_\parallel \wedge dl + d\psi \wedge d(e\vartheta + q v_\parallel).$$

- Setting:

$$\eta = e\vartheta + q v_\parallel,$$

the **canonical pairs** are $(l, mv_\parallel)$ and (η, ψ). They describe the motion parallel and perpendicular to the magnetic field respectively.

- Observe that:

$$dl\, dv_{\parallel}\, d\vartheta\, d\psi = e^{-1} dl\, dv_{\parallel}\, d\eta\, d\psi.$$

Thus, the flow $X = \left(\dot{l}, \dot{v}_{\parallel}, \dot{\vartheta}, \dot{\psi}\right)$ is **measure preserving** in virtue of Liouville's theorem [15].

- To formulate a diffusion operator on the symplectic leaf μ, consider two hypothesis:

1. **Perturbations with vanishing ensemble average** $\langle \boldsymbol{E} \rangle = \boldsymbol{0}$.

2. **Ergodic Hypothesis** [16] on the symplectic leaf:

$$\mathcal{E} = -d\varphi = \frac{m}{e} D_{\parallel}^{1/2} \Gamma_{\parallel} dl + D_{\perp}^{1/2} \Gamma_{\perp} d\vartheta + D_{\vartheta}^{1/2} \Gamma_{\vartheta} d\psi,$$

with $dW = \Gamma dt$ a Wiener process.

- The $\boldsymbol{E} \times \boldsymbol{B}$ drift velocity causing diffusion is then written as:

$$dX_{\perp} = -\frac{1}{rB} \frac{\partial \delta\varphi}{\partial \vartheta} dt = \frac{D_{\perp}^{1/2}}{rB} dW_{\perp}.$$

[15] J. W. Gibbs, *Elementary Principles in Statistical Mechanics*, chap. 1 (Scribner's Sons, New York, 1902), pp. 3-19.
[16] G. D. Birkhoff, *Proof of the Ergodic Theorem*, Proc. N. A. S. **17**, pp. 656-60 (1931).

FOKKER-PLANCK EQUATION ON THE SYMPLECTIC LEAF (I)

$$
\begin{cases}
dL = \left\{ v_\parallel + \mathfrak{C}_l + \left(\frac{1}{2} - \alpha\right) D_\perp \left[(\partial_\psi + q\partial_l) q + q(\partial_\psi + q\partial_l) \log(rB) \right] \right\} dt + q D_\perp^{1/2} dW_\perp \\
dV_\parallel = -\left(\frac{\mu}{m} \frac{\partial B}{\partial l} + \gamma v_\parallel - \mathfrak{C}_{v_\parallel} \right) dt + D_\parallel^{1/2} dW_\parallel - D_\perp^{1/2} v_\parallel q_l dW_\perp \\
d\Theta = \left[\frac{\mu}{e} (\partial_\psi + q\partial_l) B + \mathfrak{C}_\vartheta - \frac{m}{e} v_\parallel^2 q_l \right] dt - D_\vartheta^{1/2} dW_\vartheta - \frac{m}{e} q D_\parallel^{1/2} dW_\parallel \\
d\Psi = \left[D_\perp \left(\frac{1}{2} - \alpha \right) (\partial_\psi + q\partial_l) \log(rB) + \mathfrak{C}_\psi \right] dt + D_\perp^{1/2} dW_\perp
\end{cases}
$$

$$
\begin{aligned}
\frac{\partial f}{\partial t} = & -\frac{\partial}{\partial l} \left\{ v_\parallel + \mathfrak{C}_l + \left(\frac{1}{2} - \alpha\right) D_\perp \left[(\partial_\psi + q\partial_l) q + q(\partial_\psi + q\partial_l) \log(rB) \right] \right\} f \\
& + \frac{\partial}{\partial v_\parallel} \left(\frac{\mu}{m} \frac{\partial B}{\partial l} + \gamma v_\parallel - \mathfrak{C}_{v_\parallel} \right) f - \frac{\partial}{\partial \vartheta} \left[\frac{\mu}{e} (\partial_\psi + q\partial_l) B + \mathfrak{C}_\vartheta - \frac{m}{e} v_\parallel^2 q_l \right] f \\
& - \frac{\partial}{\partial \psi} \left[D_\perp \left(\frac{1}{2} - \alpha \right) (\partial_\psi + q\partial_l) \log(rB) + \mathfrak{C}_\psi \right] f + \frac{1}{2} D_\perp \frac{\partial^2}{\partial l^2} q^2 f + \frac{1}{2} D_\parallel \frac{\partial^2 f}{\partial v_\parallel^2} + \frac{1}{2} D_\perp \frac{\partial^2 f}{\partial \psi^2} + \frac{1}{2} D_\vartheta \frac{\partial^2 f}{\partial \vartheta^2} \\
& + \frac{m^2}{2e^2} D_\parallel \frac{\partial^2}{\partial \vartheta^2} q^2 f - \frac{m}{e} D_\parallel \frac{\partial^2}{\partial v_\parallel \partial \vartheta} q f + D_\perp \frac{\partial^2}{\partial l \partial \psi} q f - \alpha D_\perp \frac{\partial}{\partial l} \left[(\partial_\psi + q\partial_l) q \right] f - D_\perp \frac{\partial^2}{\partial l \partial v_\parallel} v_\parallel q q_l f \\
& + \frac{1}{2} D_\perp \frac{\partial^2}{\partial v_\parallel^2} (v_\parallel q_l)^2 f - D_\perp \frac{\partial^2}{\partial \psi \partial v_\parallel} v_\parallel q_l f + \alpha D_\perp \frac{\partial}{\partial v_\parallel} \left[v_\parallel (q\partial_l + \partial_\psi) q_l - v_\parallel q_l^2 \right] f
\end{aligned}
$$

$$
f \, dl \, d\psi \, d\vartheta \, dv_\parallel \, [d\mu \, d\vartheta_c] = fB \, dx \, dy \, dz \, dv_\parallel \, [d\mu \, d\vartheta_c]
$$

- Diffusion is controlled by D (strength of fluctuations), α (time-properties of fluctuations), and $q = -\partial_l \cdot \partial_\psi$ (non-orthogonality of tangent vectors). Approximating $D = D_\perp$, $\alpha = 1/2$, $q \sim 0$, $\partial_\vartheta = 0$, we have:

$$\frac{\partial f}{\partial t} = -v_\parallel \frac{\partial f}{\partial l} + \frac{\mu}{m}\frac{\partial B}{\partial l}\frac{\partial f}{\partial v_\parallel} + \frac{1}{2} D_\perp \frac{\partial^2 f}{\partial \psi^2}$$

- **Note**: assuming a diffusion operator arbitrarily, such as $\frac{\partial^2}{\partial x^2} D(x) f$, gives wrong results.

- The creation of the density gradient is caused by the **inhomogeneous Jacobian** B linking symplectic leaf to laboratory frame:

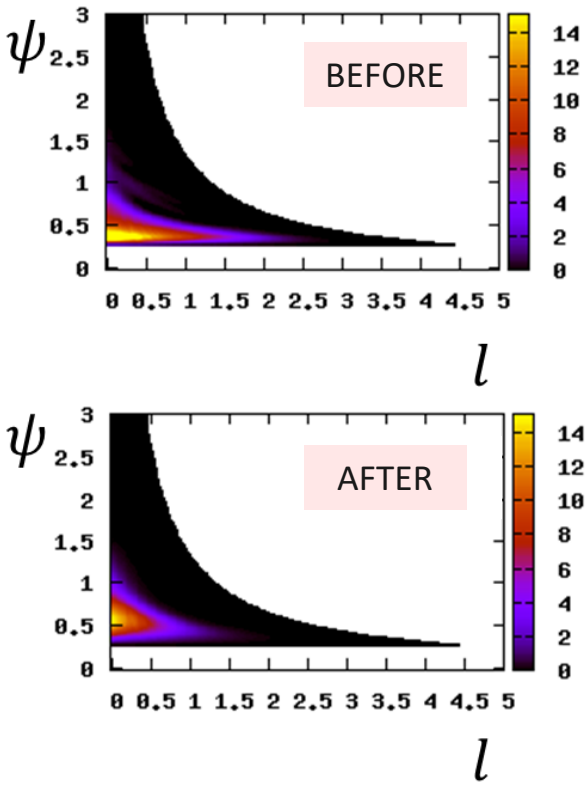
$$dl\, d\psi\, d\vartheta\, dv_\parallel = B\, dx\, dy\, dz\, dv_\parallel.$$

- The density $\rho(x, y, z)$ in the laboratory frame (x, y, z) is:

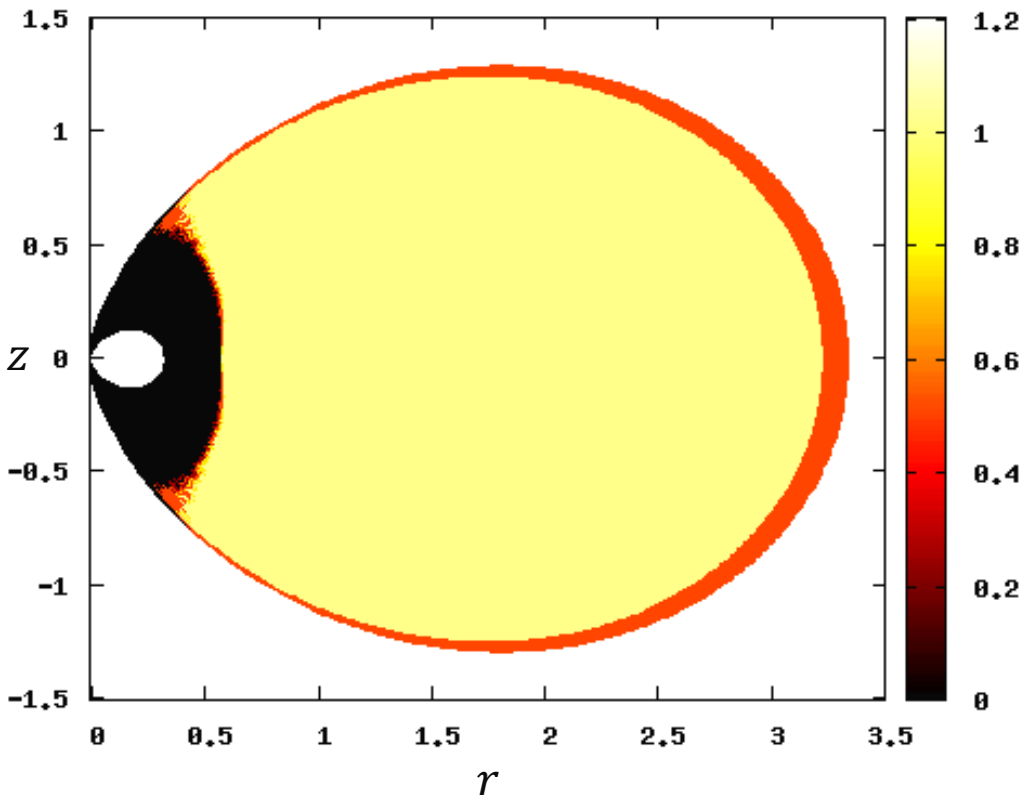
$$\rho = B\, u = B \int f\, dv_\parallel\, [d\mu\, d\vartheta_c],$$

where u is the density on the symplectic leaf. If u becomes flat due to inward diffusion, ρ must become inhomogeneous.

Symplectic Leaf
Density u

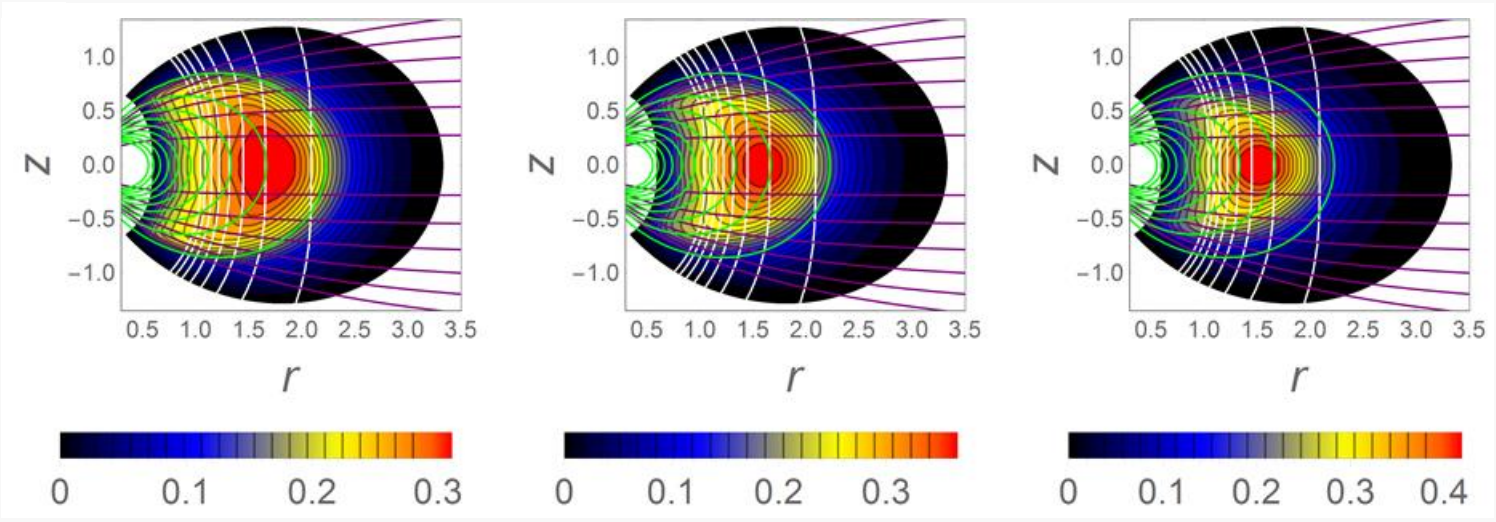


Laboratory Frame
Density $\rho = B\,u$

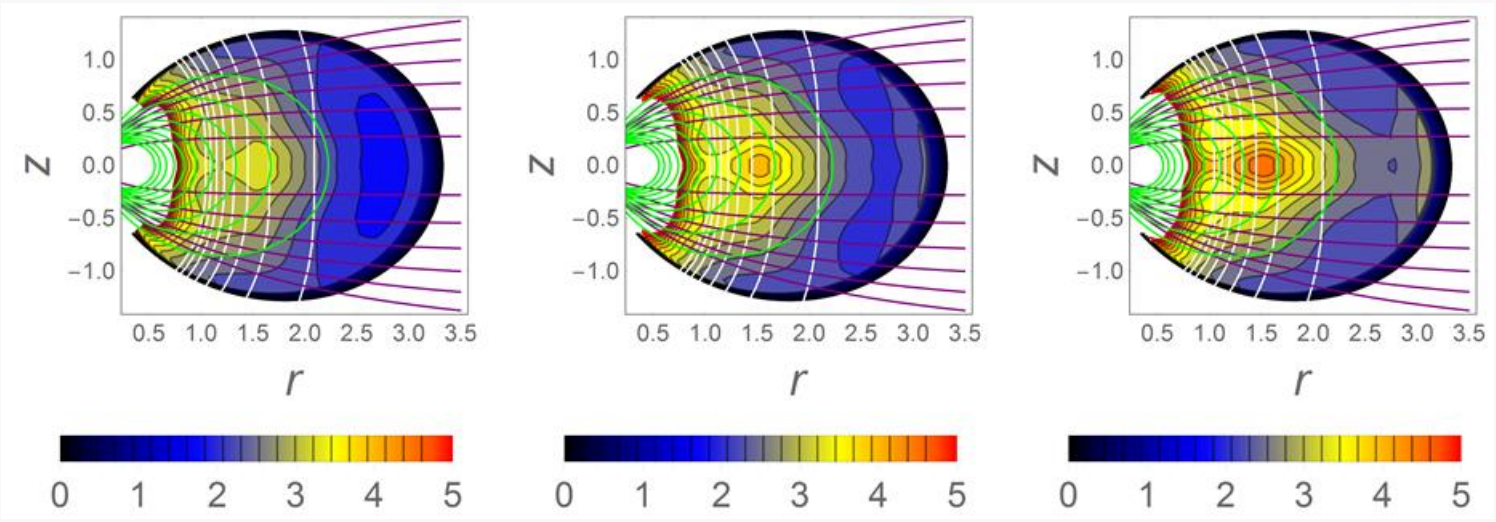


EMERGENCE OF TEMPERATURE ANISOTROPY

Laboratory Frame Density ρ



Temperature Anisotropy $T_\perp/T_\parallel$



ENTROPY (I)

- The system satisfies Boltzmann's H-theorem on the **invariant measure** [14]:

$$dV_{IM} = dl\, d\psi\, d\vartheta\, dv_{\|}\, [d\mu\, d\vartheta_c] = B\, dx\, dy\, dz\, dv_{\|}\, [d\mu\, d\vartheta_c] = B\, dV.$$

- Therefore, the **proper entropy measure** is:

$$\Sigma = -\int_{\Omega} f \log f \; dV_{IM}\,.$$

- The entropy measure in the laboratory coordinates reads:

$$\tilde{S} = -\int_{\Omega} fB \; \log(fB)\, dV = \Sigma - \langle \log B \rangle.$$

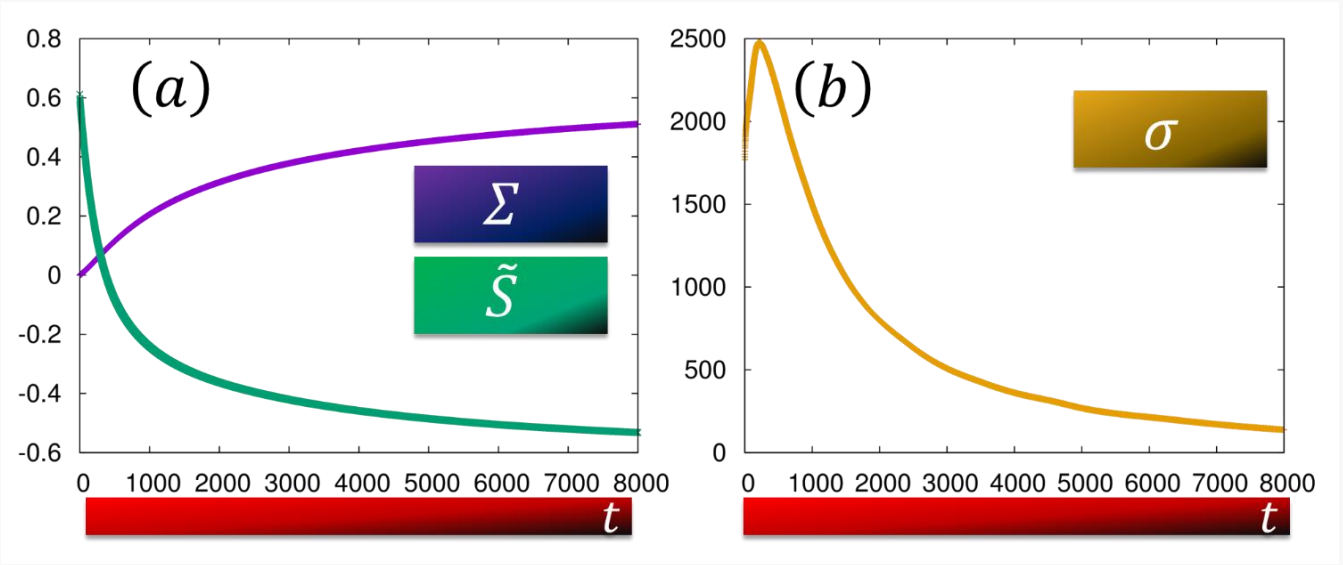
- Recalling that $\partial_t f = -\mathrm{div}(fZ)$, the rate of change in Σ and $\tilde{S}$ are:

$$\begin{cases} \dfrac{d\Sigma}{dt} = \langle \mathrm{div}(Z) \rangle - \Phi_{\partial\Omega} = \sigma - \Phi_{\partial\Omega}, \\ \dfrac{d\tilde{S}}{dt} = \langle \mathrm{div}(Z) \rangle - \Phi_{\partial\Omega} - \dfrac{d}{dt}\langle \log B \rangle = \sigma - \Phi_{\partial\Omega} - \dfrac{d}{dt}\langle \log B \rangle. \end{cases}$$

Here σ is the **entropy production rate**.

[14] N. Sato and Z. Yoshida, *Up-Hill Diffusion, Creation of Density Gradients: Entropy Measure for Systems with Topological* Constraints, Phys. Rev. E **93** 062140 (2016).

ENTROPY (II)



Pictures from [13].
(a) Time evolution of entropy measures Σ and $\tilde{S}$ during formation of a radiation belt.
(b) Time evolution of entropy production σ.

- This result can be extended to all Poisson operators $\mathcal{J}$ (and to any operator endowed with an invariant measure for any choice of H_0): the proper entropy measure Σ is defined on dV_{IM}, and the solution f to the FPE satisfies:

$$\lim_{t\to\infty} \mathcal{J}(d\log f + \beta dH_0) = 0 \qquad a.e.$$

- Darboux's theorem [14] ensures that a Poisson operator has an integrable kernel $\mathcal{J}(dC^i) = 0$. In regions $U \in \mathcal{M}$ where $\mathrm{rank}(\mathcal{J})$ is constant:

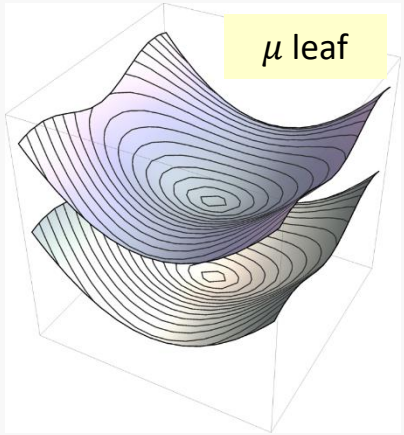
$$\lim_{t\to\infty} f = \exp\{-\beta H_0 - \gamma_i C^i\} \qquad a.e.$$

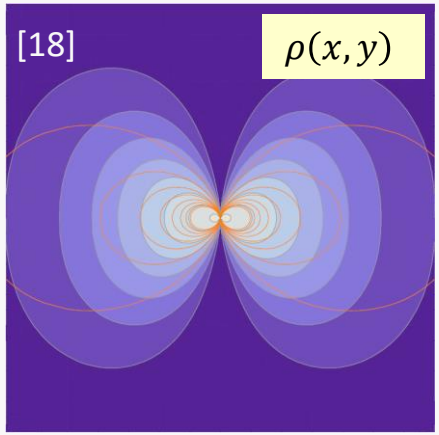
[14] N. Sato and Z. Yoshida, *Up-Hill Diffusion, Creation of Density Gradients: Entropy Measure for Systems with Topological* Constraints, Phys. Rev. E **93** 062140 (2016).
[17] M. de León and M. R. Rodrigues, *Methods of Differential Geometry in Analytical Mechanics*, Elsevier, p. 250-3 (1989).

TOPOLOGICAL CONSTRAINTS AND INTEGRABILITY

- A dipole magnetic field is integrable, i.e. it does not have helicity: $\boldsymbol{B} = \nabla\zeta \rightarrow \boldsymbol{B} \cdot \nabla \times \boldsymbol{B} = 0.$
- What happens to diffusion in an arbitrarily complex magnetic field $\boldsymbol{B} \cdot \nabla \times \boldsymbol{B} \neq 0$?

- Microscopic systems arise in Canonical Hamiltonian form.
- Macroscopic systems result from the **reduction** of negligible degrees of freedom that impart **topological constraints**.
- In non-canonical Hamiltonian systems, self-organization is driven by **integrable constraints**.





- The reduction process may destroy the Hamiltonian form: the standard formulation of statistical mechanics does not apply due to absence of phase space.
- The simplest non-Hamiltonian behavior arises in 3 dimensions:

$$\mathbf{V} = \boldsymbol{w} \times \nabla H_0.$$

[18] Z. Yoshida, *Self-Organization by Topological Constraints: Hierarchy of Foliated Phase Space*, Adv. Phys. X **1**, pp. 2-19 (2016).

$\boldsymbol{E} \times \boldsymbol{B}$ DRIFT IS NOT AN HAMILTONIAN SYSTEM

Motion of a charged particle in an e-m field

$$m\frac{d\mathbf{V}}{dt} = q(\mathbf{V} \times \boldsymbol{B} + \boldsymbol{E})$$

Canonical Hamiltonian form

- 6D phase space: $\omega^2 = dp^i \wedge dx^i = \mathcal{J}_c^{-1}$
- Charged particle Hamiltonian: $H_0 = \frac{1}{2m}(\boldsymbol{p} - q\boldsymbol{A})^2 + q\varphi$
- equations of motion: $i_{\boldsymbol{V}}\omega^2 = -dH$ with $\boldsymbol{V} = (\dot{\boldsymbol{p}}, \dot{\boldsymbol{x}})$

Reduction $m \to 0$ (non canonical Hamiltonian form)

- 3D phase space: $\omega^2 = dA_i \wedge dx^i = d\mathcal{A} = \mathfrak{B}$
- Non inertial Hamiltonian: $H_0 = q\varphi$
- Equations of motion: $i_{\mathbf{V}} d\mathcal{A} = -d\varphi$ with $\mathbf{V} \times \boldsymbol{B} + \boldsymbol{E} = \mathbf{0}$ **and** $X = (\dot{x}, \dot{y}, \dot{z})$

Reduction $\mathrm{V}_{\parallel} \to 0$

$$\mathbf{V} = \boldsymbol{w} \times \nabla H_0 = -\frac{\boldsymbol{B}}{B^2} \times \boldsymbol{E}.$$

$$\boldsymbol{w} \cdot \nabla \times \boldsymbol{w} = \frac{\boldsymbol{B} \cdot \nabla \times \boldsymbol{B}}{B^4} \neq 0 \Longrightarrow \text{violation of } \textbf{Jacobi identity}.$$

- Since $\mathbf{V} = \boldsymbol{w} \times \nabla H_0$, the motion is constrained to be orthogonal to $\boldsymbol{w}$:

$$\boldsymbol{w} \cdot \mathbf{V} = 0.$$

- Euler's equation for the motion of a rigid body with angular momentum $\boldsymbol{x}$, moment of inertia $\boldsymbol{I} = (I_x, I_y, I_z)$ and Hamiltonian $H_0 = \frac{1}{2}\left(\frac{x^2}{I_x} + \frac{y^2}{I_y} + \frac{z^2}{I_z}\right)$:

$$\mathbf{V} = \boldsymbol{x} \times \nabla H_0.$$

- This is a non-canonical Hamiltonian system with **integrable constraint** $\boldsymbol{x} \cdot \mathbf{V} = 0$ which can be integrated to the **Casimir invariant** $C = \boldsymbol{x}^2/2$:

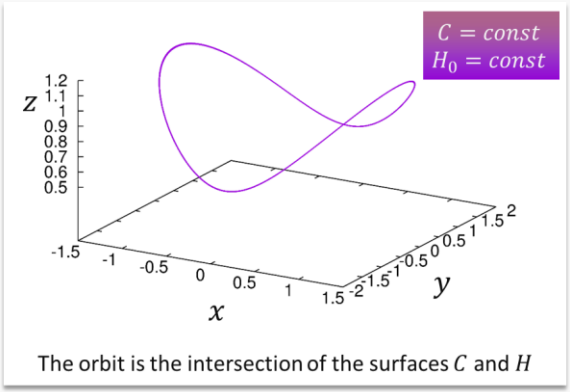
$$\dot{C} = \nabla C \cdot \mathbf{V} = \boldsymbol{x} \cdot \mathbf{V} = 0.$$

- Consider a **Beltrami field** $\boldsymbol{w} = \mu \nabla \times \boldsymbol{w}$.

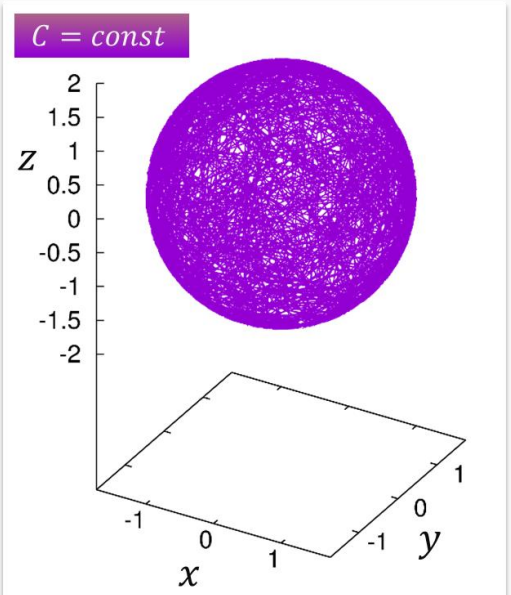
- This system has a **non-integrable constraint** since the Jacobi identity is not satisfied:

$$\boldsymbol{w} \cdot \nabla \times \boldsymbol{w} = \mu w^2 \neq 0.$$

Euler Rigid Body (Hamiltonian)
$\boldsymbol{w} = \boldsymbol{x}$

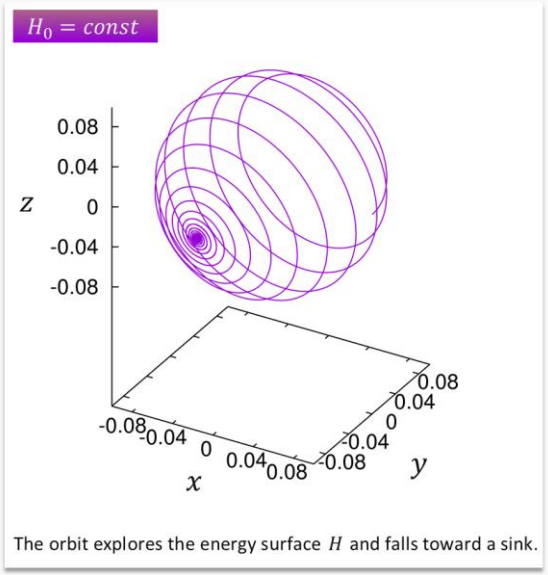


The orbit is the intersection of the surfaces C and H

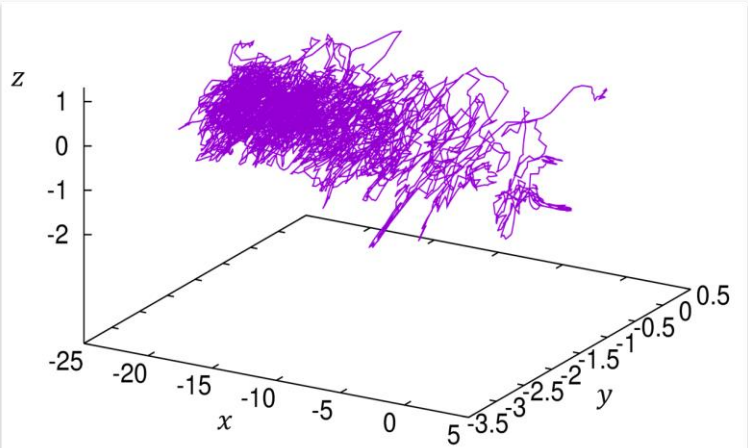


If we perturb the Hamiltonian $\nabla H = \nabla H_0 + \boldsymbol{\Gamma}$, the particle explores the surface C.

Beltrami (Non-Hamiltonian)
$\boldsymbol{w} = (\cos z - \sin y\,, -\sin z\,, \cos y)$



The orbit explores the energy surface H and falls toward a sink.



If we perturb the Hamiltonian $\nabla H = \nabla H_0 + \boldsymbol{\Gamma}$ there are no integral manifolds.

$$X = \boldsymbol{w} \times \nabla H = \boldsymbol{w} \times (\nabla H_0 + \boldsymbol{\Gamma})$$

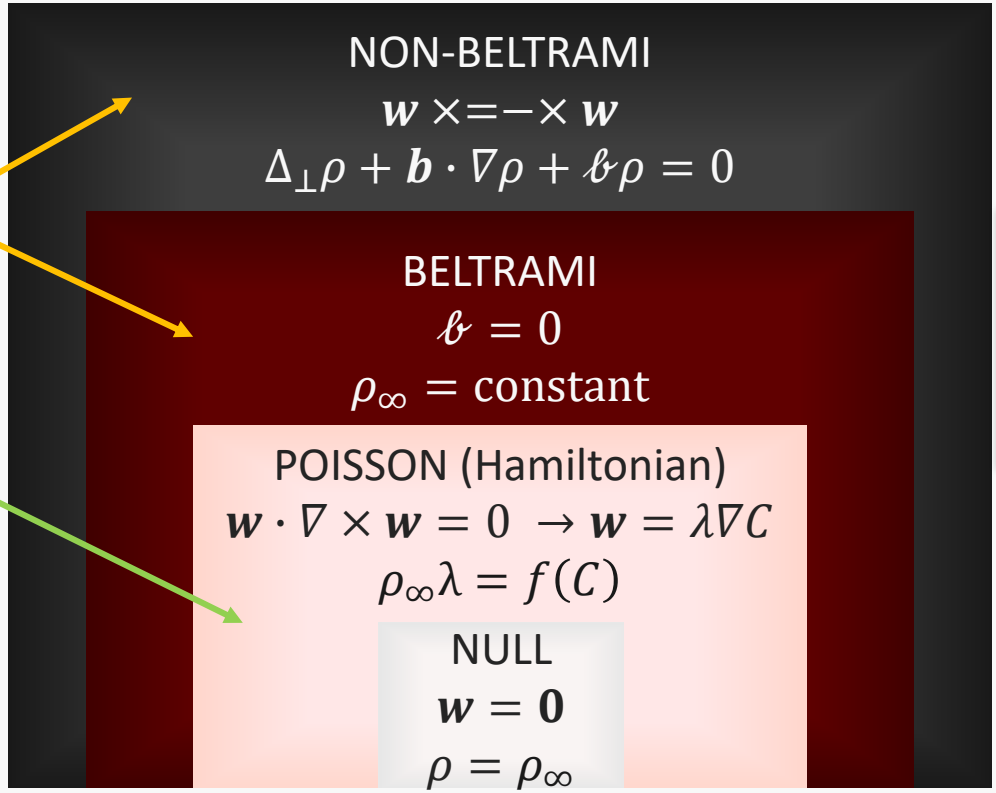
- Assume charge neutrality $\nabla H = \nabla H_0 + \boldsymbol{\Gamma} = e\nabla\varphi + \boldsymbol{\Gamma} = \boldsymbol{\Gamma}$. Then:

$$\mathbf{V} = \boldsymbol{w} \times \boldsymbol{\Gamma}, \qquad \frac{\partial \rho}{\partial t} = \frac{1}{2}\nabla \cdot [\boldsymbol{w} \times (\nabla \times \rho \boldsymbol{w})].$$

- The form of $\rho_\infty(\boldsymbol{x})$ depends on the geometrical properties of $\boldsymbol{w} = \boldsymbol{B}/B^2$.

Magnetic field with helicity $\boldsymbol{B} \cdot \nabla \times \boldsymbol{B} \neq \boldsymbol{0}$.

Magnetic field without helicity $\boldsymbol{B} \cdot \nabla \times \boldsymbol{B} = \boldsymbol{0}$.

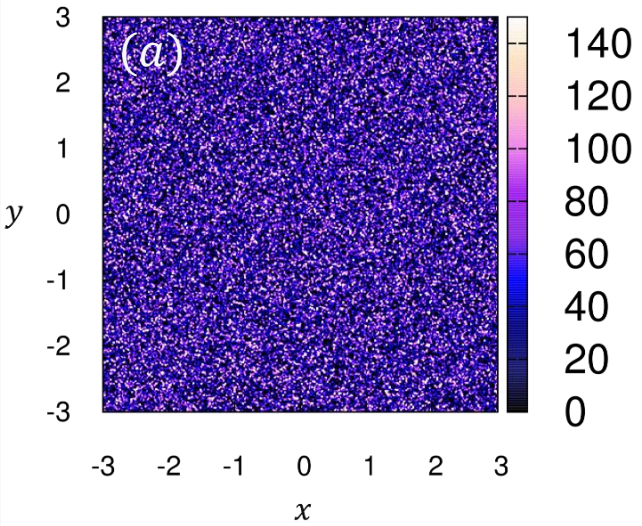


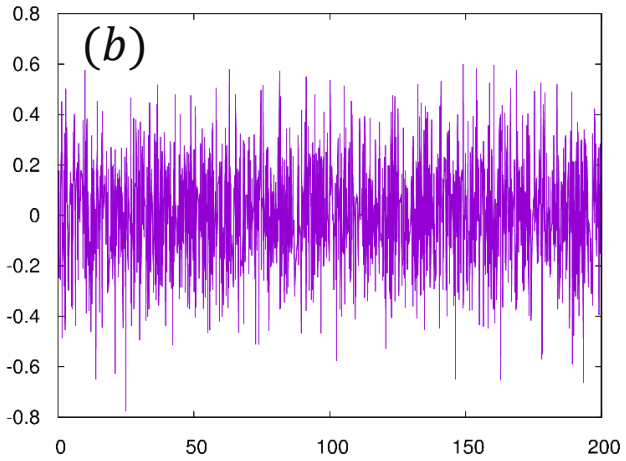
NON-BELTRAMINESS

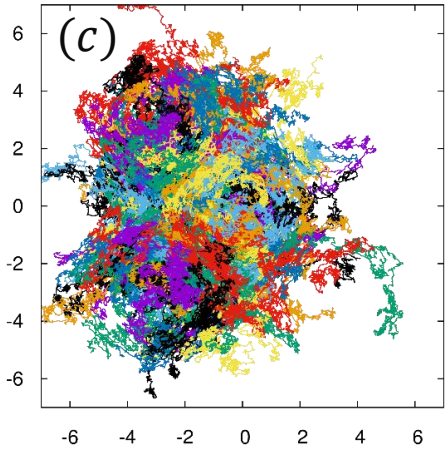
$\mathscr{b} = \nabla \cdot \boldsymbol{b}$
$= \nabla \cdot (\boldsymbol{w} \times \nabla \times \boldsymbol{w})$

NUMERICAL TEST OF 3D DIFFUSION (I)

- The theory is tested by comparing the numerical integration of the stochastic equation of motion with the analytical solution of the Fokker-Planck equation.







Sample size: $8 \cdot 10^6$ particles per simulation.

(a) Initial uniform particle distribution in 3D cube with periodic boundaries.
(b) Noise Γ_i as a function of time t.
(c) 250 sample paths in (x, y) plane for $\boldsymbol{w} = \sqrt{1 + \cos^2 x}\, \nabla(z - \cos x - \cos y)$.

NUMERICAL TEST OF 3D DIFFUSION (II)

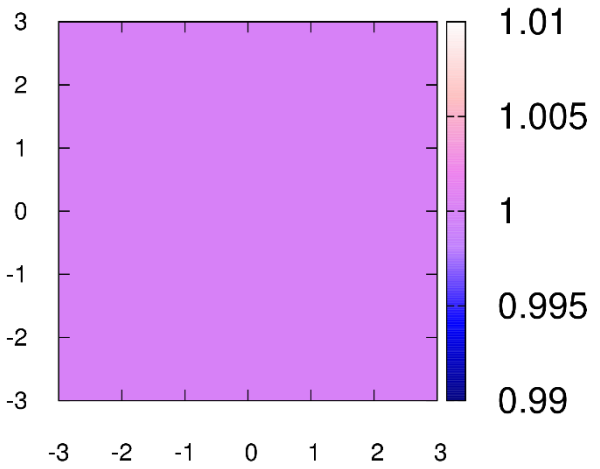
$\boldsymbol{w}$ $(\boldsymbol{B}/B^2)$	$w^{-1}(x,y)$ $(B(x,y))$	$\rho_\infty(x,y)$

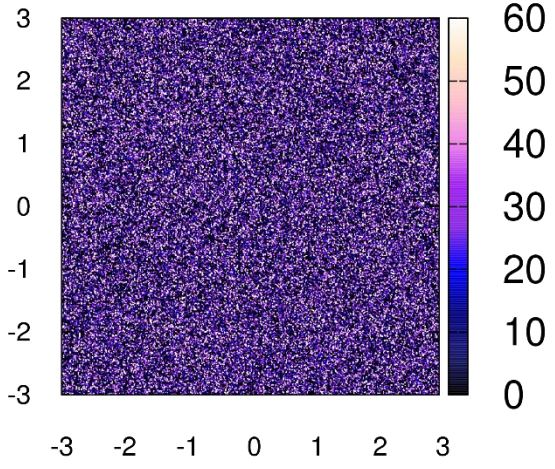
Poisson Operator
(Constant Magnetic Field)

$\boldsymbol{w} = \boldsymbol{B} = \nabla z.$

Analytical Solution

$\rho_\infty =$ constant.



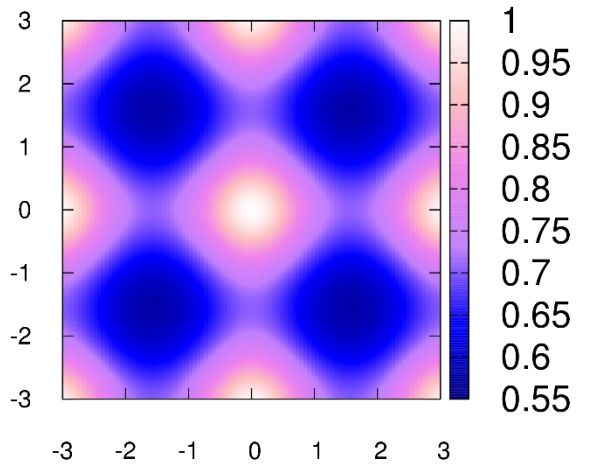


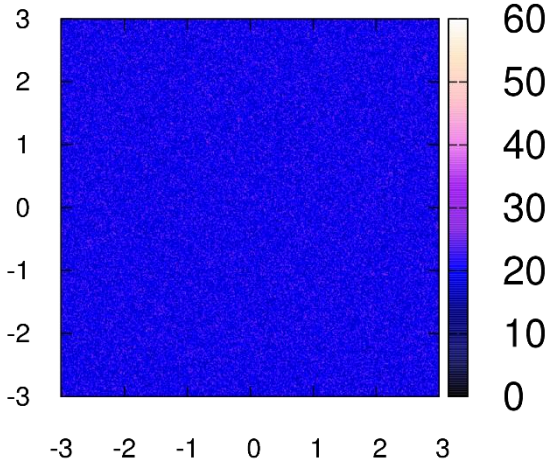
Poisson Operator
(Gradient Magnetic Field)

$\boldsymbol{w} = \nabla(z - \cos x - \cos y).$

Analytical Solution

$\rho_\infty =$ constant.





NUMERICAL TEST OF 3D DIFFUSION (III)

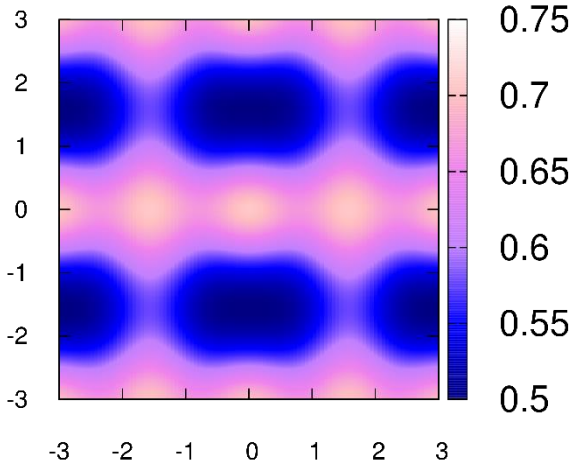
$\boldsymbol{w}$ $(\boldsymbol{B}/B^2)$	$w^{-1}(x,y)$ $(B(x,y))$	$\rho_\infty(x,y)$

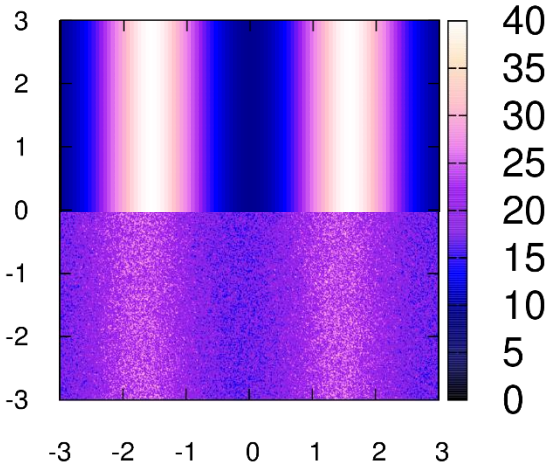
Poisson Operator
(Gradient Magnetic Field)

$$\boldsymbol{w} = \lambda \nabla C = \sqrt{1+\cos^2 x}\,\nabla(z - \cos x - \cos y).$$

Analytical Solution

$$\rho_\infty = \lambda^{-1} f(C).$$



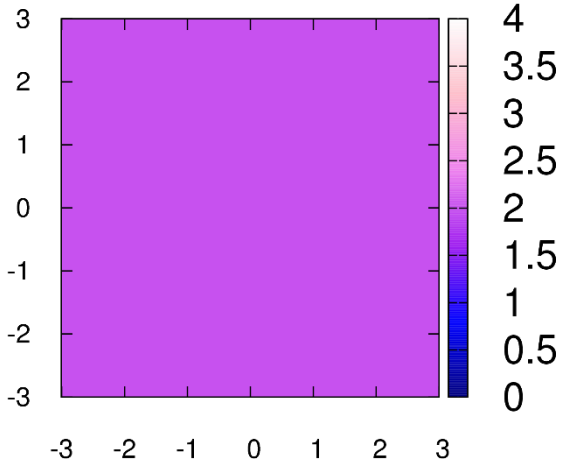


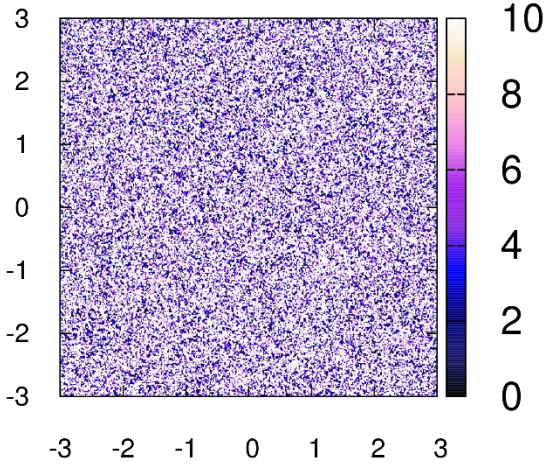
Beltrami Operator
(Beltrami Magnetic Field)

$$\boldsymbol{w} = 2\boldsymbol{B} = \begin{pmatrix} \cos z + \sin z \\ \cos z - \sin z \\ 0 \end{pmatrix}$$

Analytical Solution

$$\rho_\infty = \text{constant}.$$





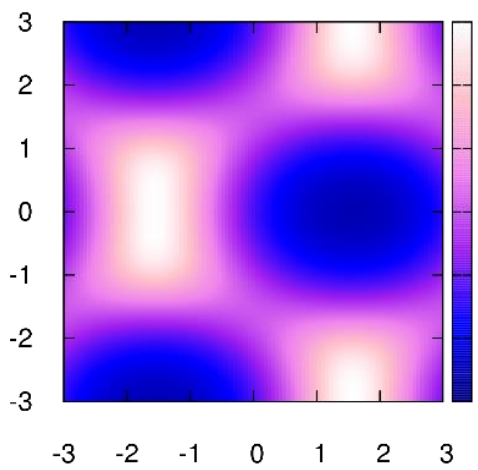
NUMERICAL TEST OF 3D DIFFUSION (IV)

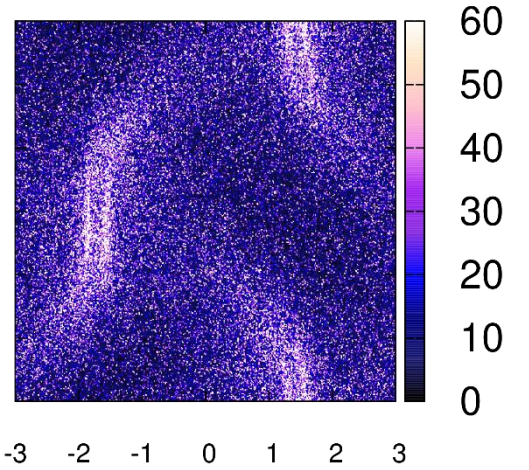
$\boldsymbol{w}$ $(\boldsymbol{B}/B^2)$	$w^{-1}(x,y)$ $(B(x,y))$	$\rho_\infty(x,y)$

Non-Beltrami Operator
(Non-Integrable Magnetic Field)

$$\mathscr{b} \neq 0$$

$$\boldsymbol{w} = (1, \sin x + \cos y\,, \cos x).$$

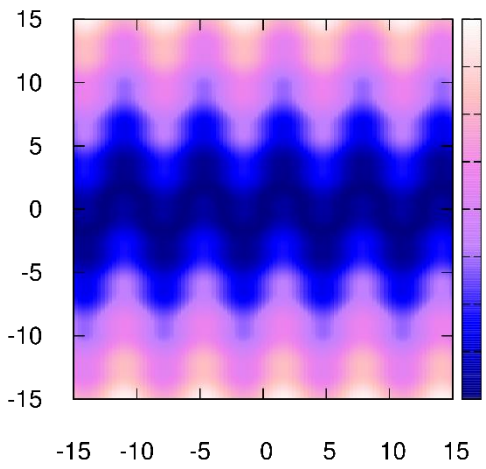


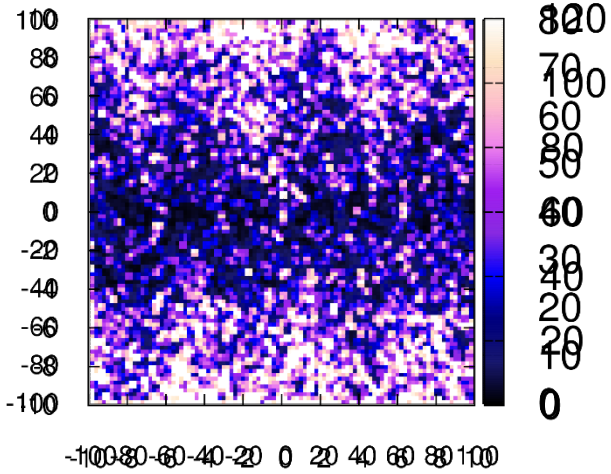


Non-Beltrami Operator
(Non-Integrable Magnetic Field)
(free boundaries)

$$\mathscr{b} \neq 0$$

$$\boldsymbol{w} = \frac{\left(1, 0, \dfrac{y - \sin y \cos y}{2} - \sin x\right)}{1 + \left(\dfrac{y - \sin y \cos y}{2} - \sin x\right)^2}.$$





- In the most general case of a non-integrable operator $\boldsymbol{w} \cdot \nabla \times \boldsymbol{w} \neq 0$ with $\mathscr{b} \neq 0$ the density profile is determined by field strength w and non-Beltraminess $\mathscr{b}$.
- To elucidate the role of $\mathscr{b}$, we consider an operator of uniform strength: $w^{-1} =$ constant.

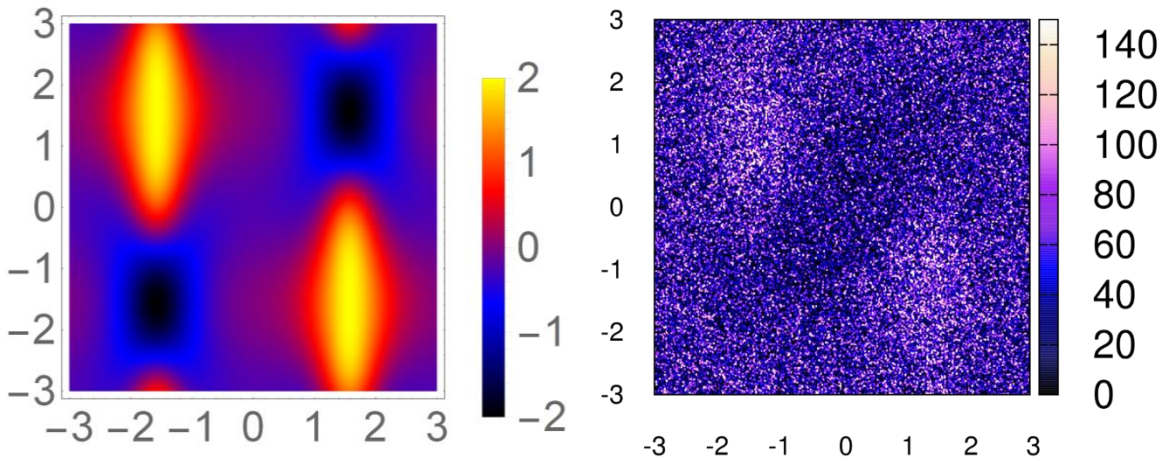
$\boldsymbol{w}$ $(\boldsymbol{B}/B^2)$	$\mathscr{b} = \nabla \cdot \boldsymbol{b}$ $= \nabla \cdot (\boldsymbol{w} \times \nabla \times \boldsymbol{w})$	$\rho_\infty(x, y)$

Non-Beltrami Operator
(Non-Integrable Magnetic Field)

$$\mathscr{b} \neq 0$$

$$\boldsymbol{w} = \frac{(\cos y\,, \cos x\,, \sin y)}{\sqrt{1 + \cos^2 x}}$$

$$w = \text{constant}$$



CONCLUSION

- Dipole fields can efficiently trap charged particles by the mechanism of inward diffusion.
- The inward diffusion occurs in virtue of the topological constraint imposed by μ.
- The creation of the density gradient is consistent with the second law of thermodynamics.

- The density profile resulting from diffusion by a white noise electric field in an arbitrary magnetic field $\boldsymbol{B}$ is determined by its geometric properties.
- In 3D, three classes of $\boldsymbol{w} = \boldsymbol{B}/B^2$ can be identified: Poisson, Beltrami, and Non-Beltrami.

POISSON (Hamiltonian)	$\boldsymbol{w} \cdot \nabla \times \boldsymbol{w} = 0$ $(\boldsymbol{w} = \lambda \nabla C)$	$\rho_\infty = \rho_\infty(\lambda, C)$
BELTRAMI	$\mathscr{b} = 0$	$\rho_\infty = \text{constant}$
NON-BELTRAMI	$\mathscr{b} \neq 0$	$\rho_\infty = \rho_\infty[w, \mathscr{b}]$

- The construction of a diffusion operator requires careful treatment, especially when the system is not endowed with a canonical Hamiltonian structure.
- The statistical theory developed here can be applied to general conservative systems.
- Diffusion by $\boldsymbol{E} \times \boldsymbol{B}$ drift is of special mathematical interest because it involves a non-elliptic partial differential equation.

End