Hamiltonian and Non-Hamiltonian Reductions of Charged Particle Dynamics: Diffusion and Self-Organization

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Magnetospheres are ubiquitous in the universe.

Dipole magnetic fields confine charged particles.

Spontaneous confinement (*self-organization*) is achieved by *inward diffusion*.
MAGNETOSPHERIC CONFINEMENT OF CHARGED PARTICLES

- Magnetospheric confinement is a promising source of applications.
- Advanced fusion reactor based on $D - ^{3}H$ reaction [1].
- Trap for matter and antimatter plasmas [2].
- Study of astrophysical systems [3].

- The creation of the density gradient seemingly violates the entropy principle.

OUTLINE

1. Diffusion in a dipole field: correct expression of diffusion operator in magnetic coordinates.
2. Diffusion in non-integrable magnetic field $\mathbf{B} \cdot \nabla \times \mathbf{B} \neq 0$.

- RT-1 produces a laboratory magnetosphere.
- A similar device is LDX at MIT.

First Plasma Experiment on RT-1
EXPERIMENTAL OBSERVATIONS

MAGNETIC FIELD

\[ \mathbf{B} = \nabla \psi \times \nabla \theta \]

FLUX FUNCTION

\[ \psi(r, z) = \frac{r^2}{(r^2 + z^2)^{3/2}} \]

LENGTH OF A FIELD LINE [7]

\[ l(r, z) = \frac{1}{2\psi} \left[ \frac{1}{\sqrt{3}} \log \left( \sqrt{3} \sqrt{1 - (r\psi)^{2/3}} \right) \right] \]
It is thought that particle number per flux-tube volume tends to be homogenized [9,10].

In a dipole plasma, the key normal transport mechanism is diffusion by $E \times B$ drift:

$$v_\perp = \frac{E_\perp}{B}.$$
• Conservation of adiabatic invariants foliates phase space: particle motion is constrained on leaves [11,12].

• Electromagnetic fluctuations break the weakest topological constraints. The result is a random process among leaves.

• A diffusion operator must be formulated on the corresponding invariant measure [8,13].

• Entropy should be maximized on the symplectic leaf [14].

The constraint causing non-canonicality is the conservation of $\mu$: the pair $(\vartheta_c, \mu)$ can be **reduced**:

$$dx \, dy \, dz \, dp_x \, dp_y \, dp_z = m^2 dl \, dv_{||} \, d\vartheta \, d\psi \, [d\vartheta_c \, d\mu].$$

Dynamical variables are reduced to four: $(l, v_{||}, \vartheta, \psi)$.

The reduced equations are the guiding center equations:

\[
\begin{align*}
\dot{v}_{||} &= -m^{-1}(\mu B + e\varphi)_l + v_{||}v_{E \times B} \cdot k, \\
v &= v_{||} + v_{E \times B} + v_{\nabla B} + v_k.
\end{align*}
\]

Transforming to the new coordinates:

\[
\begin{align*}
i &= v_{||} - q\varphi_{\vartheta}, \\
\dot{v}_{||} &= -m^{-1}(\mu B + e\varphi)_l + v_{||}q_l \varphi_{\vartheta}, \\
\dot{\vartheta} &= (\partial_{\psi} + q \partial_l)(e^{-1}\mu B + \varphi) - e^{-1}mv_{||}^2 q_l, \\
\dot{\psi} &= -\varphi_{\vartheta}.
\end{align*}
\]
CONSTRUCTION OF CANONICAL COORDINATES

• The reduced system should be Hamiltonian because the reduced variable $\mu$ is a dynamical constant and the phase $\vartheta_c$ can be neglected.

• We look for the nondegenerate closed two-form $\omega = d\lambda$ such that:

$$i_X \omega = -dH_0,$$

with $H_0 = \mu B + \frac{m}{2} v_\parallel^2 + e\varphi$ the Hamiltonian function. Setting $q = -\partial_l \cdot \partial_\psi$:

$$\omega = mdv_\parallel \wedge dl + d\psi \wedge d(e\vartheta + qv_\parallel).$$

• Setting:

$$\eta = e\vartheta + qv_\parallel,$$

the canonical pairs are $(l, mv_\parallel)$ and $(\eta, \psi)$. They describe the motion parallel and perpendicular to the magnetic field respectively.
• Observe that:

\[ dl \, dv_\parallel \, d\vartheta \, d\psi = e^{-1} dl \, dv_\parallel \, d\eta \, d\psi. \]

Thus, the flow \( X = (\hat{l}, \hat{v}_\parallel, \dot{\vartheta}, \dot{\psi}) \) is **measure preserving** in virtue of Liouville’s theorem [15].

• To formulate a diffusion operator on the symplectic leaf \( \mu \), consider two hypothesis:

1. **Perturbations with vanishing ensemble average** \( \langle E \rangle = 0 \).

2. **Ergodic Hypothesis** [16] on the symplectic leaf:

\[ \mathcal{E} = -d\varphi = \frac{m}{e} D_\parallel^{1/2} \Gamma_\parallel dl + D_\perp^{1/2} \Gamma_\perp d\vartheta + D_\vartheta^{1/2} \Gamma_\vartheta d\psi, \]

with \( dW = \Gamma dt \) a Wiener process.

• The \( E \times B \) drift velocity causing diffusion is then written as:

\[ dX_\perp = -\frac{1}{rB} \frac{\partial \delta \varphi}{\partial \vartheta} dt = \frac{D_\perp^{1/2}}{rB} dW_\perp. \]

---


\[
\begin{align*}
dL &= \left\{ v_\parallel + C_t + \left( \frac{1}{2} - \alpha \right) D_\perp \left[ (\partial_\psi + q \partial_\lambda) q + q (\partial_\psi + q \partial_\lambda) \log(rB) \right] \right\} dt + q D_\perp^{1/2} dW_\perp \\
dV_\parallel &= - \left( \frac{\mu \partial B}{m \partial l} + \gamma v_\parallel - C_{v_\parallel} \right) dt + D_\parallel^{1/2} dW_\parallel - D_\perp^{1/2} v_\parallel q_\lambda dW_\perp \\
d\theta &= \left[ \frac{\mu}{e} (\partial_\psi + q \partial_\lambda) B + C_\theta - \frac{m}{e} v_\parallel^2 q_\lambda \right] dt - D_\theta^{1/2} dW_\theta - \frac{m}{e} q D_\parallel^{1/2} dW_\parallel \\
d\psi &= \left[ D_\perp \left( \frac{1}{2} - \alpha \right) (\partial_\psi + q \partial_\lambda) \log(rB) + C_\psi \right] dt + D_\perp^{1/2} dW_\perp
\end{align*}
\]

\[
\begin{align*}
\frac{\partial f}{\partial t} &= - \frac{\partial}{\partial l} \left\{ v_\parallel + C_t + \left( \frac{1}{2} - \alpha \right) D_\perp \left[ (\partial_\psi + q \partial_\lambda) q + q (\partial_\psi + q \partial_\lambda) \log(rB) \right] \right\} f \\
&\quad + \frac{\partial}{\partial v_\parallel} \left( \frac{\mu}{m} \frac{\partial B}{\partial l} + \gamma v_\parallel - C_{v_\parallel} \right) f - \frac{\partial}{\partial \theta} \left[ \frac{\mu}{e} (\partial_\psi + q \partial_\lambda) B + C_\theta - \frac{m}{e} v_\parallel^2 q_\lambda \right] f \\
&\quad - \frac{\partial}{\partial \psi} \left[ D_\perp \left( \frac{1}{2} - \alpha \right) (\partial_\psi + q \partial_\lambda) \log(rB) + C_\psi \right] f + \frac{1}{2} D_\perp \frac{\partial^2}{\partial l^2} q^2 f + \frac{1}{2} D_\parallel \frac{\partial^2}{\partial v_\parallel^2} f + \frac{1}{2} D_\perp \frac{\partial^2}{\partial \theta^2} f + \frac{1}{2} D_\theta \frac{\partial^2}{\partial \theta^2} f \\
&\quad + \frac{m^2}{2e^2} D_\parallel \frac{\partial^2}{\partial \theta^2} q f - \frac{m}{e} D_\parallel \frac{\partial^2}{\partial v_\parallel \partial \theta} q f + D_\perp \frac{\partial^2}{\partial l \partial \psi} q f - \alpha D_\perp \frac{\partial}{\partial l} \left[ (\partial_\psi + q \partial_\lambda) q f - D_\perp \frac{\partial^2}{\partial l \partial v_\parallel} v_\parallel q_\lambda f \right] f \\
&\quad + \frac{1}{2} D_\perp \frac{\partial^2}{\partial v_\parallel^2} \left( v_\parallel q_\lambda \right) f - D_\perp \frac{\partial^2}{\partial \psi \partial v_\parallel} v_\parallel q_\lambda f + \alpha D_\perp \frac{\partial}{\partial v_\parallel} \left[ v_\parallel (q \partial_\lambda + \partial_\psi) q_\lambda - v_\parallel q_\lambda^2 \right] f
\end{align*}
\]

\[
f \ d\lambda \ d\psi \ d\theta \ d\psi_c \left[ d\mu \ d\theta_c \right] = fB \ d\chi \ d\psi \ d\chi_c \ d\psi_c \left[ d\mu \ d\theta \right]
\]
• Diffusion is controlled by $D$ (strength of fluctuations), $\alpha$ (time-properties of fluctuations), and $q = -\partial_l \cdot \partial_\psi$ (non-orthogonality of tangent vectors). Approximating $D = D_\perp$, $\alpha = 1/2$, $q \sim 0$, $\partial_{\psi} = 0$, we have:

$$\frac{\partial f}{\partial t} = -v_\parallel \frac{\partial f}{\partial l} + \frac{\mu}{m} \frac{\partial B}{\partial l} \frac{\partial f}{\partial v_\parallel} + \frac{1}{2} D_\perp \frac{\partial^2 f}{\partial \psi^2}$$

• **Note:** assuming a diffusion operator arbitrarily, such as $\frac{\partial^2}{\partial x^2} D(x) f$, gives wrong results.

• The creation of the density gradient is caused by the **inhomogeneous Jacobian** $B$ linking symplectic leaf to laboratory frame:

$$dl \; d\psi \; d\theta \; dv_\parallel = B \; dx \; dy \; dz \; dv_\parallel.$$  

• The density $\rho(x, y, z)$ in the laboratory frame $(x, y, z)$ is:

$$\rho = B \; u = B \int f \; dv_\parallel \; [d\mu \; d\theta_c],$$

where $u$ is the density on the symplectic leaf. If $u$ becomes flat due to inward diffusion, $\rho$ must become inhomogeneous.
CREATION OF DENSITY GRADIENT

Symplectic Leaf Density $u$

Laboratory Frame Density $\rho = B \ u$

BEFORE

AFTER
EMERGENCE OF TEMPERATURE ANISOTROPY

Laboratory Frame Density $\rho$

Temperature Anisotropy $T_\perp/T_\parallel$
The system satisfies Boltzmann’s H-theorem on the invariant measure [14]:

\[ dV_{IM} = dl \, d\psi \, d\vartheta \, dv_{\parallel} \, [d\mu \, d\vartheta_c] = B \, dx \, dy \, dz \, dv_{\parallel} \, [d\mu \, d\vartheta_c] = B \, dV. \]

Therefore, the proper entropy measure is:

\[ \Sigma = - \int_{\Omega} f \log f \, dV_{IM}. \]

The entropy measure in the laboratory coordinates reads:

\[ \tilde{S} = - \int_{\Omega} fB \, \log(fB) \, dV = \Sigma - \langle \log B \rangle. \]

Recalling that \( \partial_t f = -\text{div}(fZ) \), the rate of change in \( \Sigma \) and \( \tilde{S} \) are:

\[
\begin{align*}
\frac{d\Sigma}{dt} &= \langle \text{div}(Z) \rangle - \Phi_{\partial\Omega} = \sigma - \Phi_{\partial\Omega}, \\
\frac{d\tilde{S}}{dt} &= \langle \text{div}(Z) \rangle - \Phi_{\partial\Omega} - \frac{d}{dt} \langle \log B \rangle = \sigma - \Phi_{\partial\Omega} - \frac{d}{dt} \langle \log B \rangle.
\end{align*}
\]

Here \( \sigma \) is the entropy production rate.

This result can be extended to all Poisson operators $\mathcal{J}$ (and to any operator endowed with an invariant measure for any choice of $H_0$): the proper entropy measure $\Sigma$ is defined on $dV_{IM}$, and the solution $f$ to the FPE satisfies:

$$\lim_{t \to \infty} \mathcal{J}(d \log f + \beta dH_0) = 0 \quad a.e.$$ 

Darboux’s theorem [14] ensures that a Poisson operator has an integrable kernel $\mathcal{J}(dC^i) = 0$. In regions $U \in \mathcal{M}$ where $\text{rank}(\mathcal{J})$ is constant:

$$\lim_{t \to \infty} f = \exp\{-\beta H_0 - \gamma_i C^i\} \quad a.e.$$ 


• A dipole magnetic field is integrable, i.e. it does not have helicity: $\mathbf{B} = \nabla \zeta \rightarrow \mathbf{B} \cdot \nabla \times \mathbf{B} = 0$.

• What happens to diffusion in an arbitrarily complex magnetic field $\mathbf{B} \cdot \nabla \times \mathbf{B} \neq 0$?

• Microscopic systems arise in Canonical Hamiltonian form.

• Macroscopic systems result from the reduction of negligible degrees of freedom that impart topological constraints.

• In non-canonical Hamiltonian systems, self-organization is driven by integrable constraints.

• The reduction process may destroy the Hamiltonian form: the standard formulation of statistical mechanics does not apply due to absence of phase space.

• The simplest non-Hamiltonian behavior arises in 3 dimensions:

\[ \mathbf{V} = \mathbf{w} \times \nabla H_0. \]

**$E \times B$ DRIFT IS NOT AN HAMILTONIAN SYSTEM**

Motion of a charged particle in an e-m field

$$m \frac{dV}{dt} = q(V \times B + E)$$

**Canonical Hamiltonian form**

- 6D phase space: $\omega^2 = dp^i \wedge dx^i = J_c^{-1}$
- Charged particle Hamiltonian: $H_0 = \frac{1}{2m} (p - qA)^2 + q\varphi$
- Equations of motion: $i_V \omega^2 = -dH$ with $V = (\dot{p}, \dot{x})$

**Reduction** $m \to 0$ (non canonical Hamiltonian form)

- 3D phase space: $\omega^2 = dA_i \wedge dx^i = dA = B$
- Non inertial Hamiltonian: $H_0 = q\varphi$
- Equations of motion: $i_V dA = -d\varphi$ with $V \times B + E = 0$ and $X = (\dot{x}, \dot{y}, \dot{z})$

**Reduction** $V_\| \to 0$

$$V = w \times \nabla H_0 = -\frac{B}{B^2} \times E.$$ 

$$w \cdot \nabla \times w = \frac{B \cdot \nabla \times B}{B^4} \neq 0 \implies \text{violation of Jacobi identity.}$$
• Since \( \mathbf{V} = \mathbf{w} \times \nabla H_0 \), the motion is constrained to be orthogonal to \( \mathbf{w} \):

\[
\mathbf{w} \cdot \mathbf{V} = 0.
\]

• Euler’s equation for the motion of a rigid body with angular momentum \( \mathbf{x} \), moment of inertia \( \mathbf{I} = (I_x, I_y, I_z) \) and Hamiltonian \( H_0 = \frac{1}{2} \left( \frac{x^2}{I_x} + \frac{y^2}{I_y} + \frac{z^2}{I_z} \right) \):

\[
\mathbf{V} = \mathbf{x} \times \nabla H_0.
\]

• This is a non-canonical Hamiltonian system with integrable constraint \( \mathbf{x} \cdot \mathbf{V} = 0 \) which can be integrated to the Casimir invariant \( C = \frac{x^2}{2} \):

\[
\dot{C} = \nabla C \cdot \mathbf{V} = \mathbf{x} \cdot \mathbf{V} = 0.
\]

• Consider a Beltrami field \( \mathbf{w} = \mu \nabla \times \mathbf{w} \).

• This system has a non-integrable constraint since the Jacobi identity is not satisfied:

\[
\mathbf{w} \cdot \nabla \times \mathbf{w} = \mu \mathbf{w}^2 \neq 0.
\]
**CONSTRAINED ORBITS**

**Euler Rigid Body (Hamiltonian)**
\[ \mathbf{w} = x \]

The orbit is the intersection of the surfaces \( C \) and \( H \)

**Beltrami (Non-Hamiltonian)**
\[ \mathbf{w} = (\cos z - \sin y, -\sin z, \cos y) \]

The orbit explores the energy surface \( H \) and falls toward a sink.

\[ X = \mathbf{w} \times \nabla H = \mathbf{w} \times (\nabla H_0 + \Gamma) \]
Assume charge neutrality $\nabla H = \nabla H_0 + \Gamma = e\nabla \varphi + \Gamma = \Gamma$. Then:

$$V = w \times \Gamma, \quad \frac{\partial \rho}{\partial t} = \frac{1}{2} \nabla \cdot [w \times (\nabla \times \rho w)].$$

The form of $\rho_\infty(x)$ depends on the geometrical properties of $w = B/B^2$.
The theory is tested by comparing the numerical integration of the stochastic equation of motion with the analytical solution of the Fokker-Planck equation.

Sample size: $8 \cdot 10^6$ particles per simulation.

(a) Initial uniform particle distribution in 3D cube with periodic boundaries.
(b) Noise $L_t$ as a function of time $t$.
(c) 250 sample paths in $(x, y)$ plane for $w = \sqrt{1 + \cos^2 x} \nabla (z - \cos x - \cos y)$. 
NUMERICAL TEST OF 3D DIFFUSION (II)

**Poisson Operator (Constant Magnetic Field)**

\[ w = B = \nabla z. \]

**Analytical Solution**

\[ \rho_\infty = \text{constant}. \]

**Poisson Operator (Gradient Magnetic Field)**

\[ w = \nabla (z - \cos x - \cos y). \]

**Analytical Solution**

\[ \rho_\infty = \text{constant}. \]
**Poisson Operator**  
(Gradient Magnetic Field)  
\[ \mathbf{w} = \lambda \nabla C = \sqrt{1 + \cos^2 x} \]  
\[ \nabla (z - \cos x - \cos y). \]  

**Analytical Solution**  
\[ \rho_{\infty} = \lambda^{-1} f(C). \]

**Beltrami Operator**  
(Beltrami Magnetic Field)  
\[ \mathbf{w} = 2B = \begin{pmatrix} \cos z + \sin z \\ \cos z - \sin z \\ 0 \end{pmatrix} \]  

**Analytical Solution**  
\[ \rho_{\infty} = \text{constant}. \]
Non-Beltrami Operator
(Non-Integrable Magnetic Field)

\[ \mathcal{B} \neq 0 \]
\[ w = (1, \sin x + \cos y, \cos x). \]

Non-Beltrami Operator
(Non-Integrable Magnetic Field)
(free boundaries)

\[ \mathcal{B} \neq 0 \]
\[ w = \left(1, \frac{y - \sin y \cos y}{2} - \sin x \right) \]
\[ 1 + \left(\frac{y - \sin y \cos y}{2} - \sin x \right)^2. \]
• In the most general case of a non-integrable operator $w \cdot \nabla \times w \neq 0$ with $\mathcal{B} \neq 0$ the density profile is determined by field strength $w$ and non-Beltraminess $\mathcal{B}$.

• To elucidate the role of $\mathcal{B}$, we consider an operator of uniform strength: $w^{-1} = \text{constant}$.

### Non-Beltrami Operator

(Non-Integrable Magnetic Field)

$$w = \frac{(\cos y, \cos x, \sin y)}{\sqrt{1 + \cos^2 x}}$$

$w = \text{constant}$
CONCLUSION

• Dipole fields can efficiently trap charged particles by the mechanism of inward diffusion.

• The inward diffusion occurs in virtue of the topological constraint imposed by \( \mu \).

• The creation of the density gradient is consistent with the second law of thermodynamics.

• The density profile resulting from diffusion by a white noise electric field in an arbitrary magnetic field \( B \) is determined by its geometric properties.

• In 3D, three classes of \( w = B/B^2 \) can be identified: Poisson, Beltrami, and Non-Beltrami.

<table>
<thead>
<tr>
<th>POISSON (Hamiltonian)</th>
<th>( w \cdot \nabla \times w = 0 ) ( (w = \lambda \nabla C) )</th>
<th>( \rho_\infty = \rho_\infty (\lambda, C) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>BELTRAMI</td>
<td>( \mathcal{B} = 0 )</td>
<td>( \rho_\infty = \text{constant} )</td>
</tr>
<tr>
<td>NON-BELTRAMI</td>
<td>( \mathcal{B} \neq 0 )</td>
<td>( \rho_\infty = \rho_\infty [w, \mathcal{B}] )</td>
</tr>
</tbody>
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• The construction of a diffusion operator requires careful treatment, especially when the system is not endowed with a canonical Hamiltonian structure.

• The statistical theory developed here can be applied to general conservative systems.

• Diffusion by \( E \times B \) drift is of special mathematical interest because it involves a non-elliptic partial differential equation.