7-20 For the ground state of the hydrogen atom, find the probability of finding the electron in the range $\Delta r = 0.03a_0$ at (a) $r = a_0$ and at (b) $r = 2a_0$.

From equations 7-30 and 7-31, the wavefunction for hydrogen in the ground state is

$$\psi_{100} = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a_0}\right)^{3/2} e^{-r/a_0}$$

And the probability of finding the electron in a spherical shell of thickness dr (or Δr) is...

$$P(r)\,dr = \psi^*\psi\, 4\pi r^2 dr = \frac{4\pi r^2}{\pi}\cdot\frac{1}{a_0^3}\, e^{-2r/a_0}\, dr$$

As long as Δr is small compared to the distance over which the probability varies ... that is $\Delta r \ll a_0$ ... then the probability is...

$$P(r)\Delta r = 4\left(\frac{r}{a_0}\right)^2 e^{-2r/a_0}\cdot\left(\frac{\Delta r}{a_0}\right)$$

dimensionless, as it should be...

(a) For $\Delta r = 0.03a_0$ and at $r = a_0$...

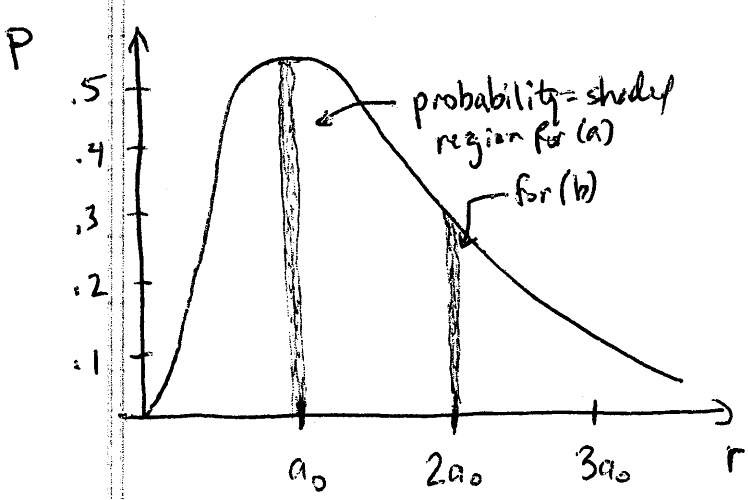
$$P(a_0)\cdot\Delta r = 4(1)^2\cdot e^{-2}\cdot 0.03 = 0.0162$$

$$= \underline{\underline{1.62\%}}$$

(b) For $\Delta r = 0.03a_0$ at $r = 2a_0$..

$$P(2a_0)\Delta r = 4(2)^2 e^{-4}\cdot 0.03 = 0.00879$$

$$= \underline{\underline{0.88\%}}$$



$\boxed{7\text{-}25}$ Hydrogen in the 2S state has wavefunction with normalization factor

$$C_{200} = \frac{1}{\sqrt{2\pi}}\left(\frac{Z}{a_0}\right)^{3/2}$$

NOTE: This is an error in the text... it should be

$$C_{200} = \frac{1}{4}\frac{1}{\sqrt{2\pi}}\left(\frac{Z}{a_0}\right)^{3/2}$$

Using value in text | **Using corrected value**

(a) The wavefunction is

Using value in text:

$$\psi_{200} = \frac{1}{\sqrt{2\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\left(2-\frac{Zr}{a_0}\right)e^{-Zr/2a_0}$$

For hydrogen $Z=1$, and at $r=a_0$...

$$\psi_{200} = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{a_0}\right)^{3/2}(2-1)e^{-1}$$

$$= 0.147\,\frac{1}{a_0^{3/2}}$$

Using corrected value:

$$\psi_{200} = \frac{1}{4}\frac{1}{\sqrt{2\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\left(2-\frac{Zr}{a_0}\right)e^{-Zr/2a_0}$$

$$= 0.0367\,\frac{1}{a_0^{3/2}}$$

(b) The square of the wavefunction is

Using value in text:

$$|\psi|^2_{200} = \frac{1}{2\pi}\left(\frac{Z}{a_0}\right)^3\left(2-\frac{Zr}{a_0}\right)^2 e^{-2Zr/a_0}$$

For hydrogen and at $r=a_0$..

$$|\psi|^2_{200} = \frac{1}{2\pi}\left(\frac{1}{a_0}\right)^3(2-1)^2 e^{-2}$$

$$= 0.0215\,\frac{1}{a_0^3}$$

Using corrected value:

$$|\psi|^2_{200} = \frac{1}{32\pi}\left(\frac{Z}{a_0}\right)^3\left(2-\frac{Zr}{a_0}\right)^2 e^{-2Zr/a_0}$$

$$|\psi_{200}|^2 = \frac{1}{32\pi}\left(\frac{1}{a_0}\right)^3(2-1)^2 e^{-2}$$

$$= 0.00135\,\frac{1}{a_0^3}$$

(c) The radial probability density is.. $P(r) = |\psi|^2 4\pi r^2$ (Eq. 7-32), so...

$$P(r) = \frac{4\pi}{2\pi}\left(\frac{Z}{a_0}\right)^3 \cdot r^2\left(2-\frac{Zr}{a_0}\right)^2 e^{-2Zr/a_0}$$

$$= 2\left(\frac{r}{a_0}\right)^2\left(2-\frac{r}{a_0}\right)^2 e^{-2r/a_0}\,\frac{1}{a_0} = 2\cdot 1^2\cdot 1^2\cdot e^{-2} = 0.271\,\frac{1}{a_0}$$

Using corrected value:

$$= 0.0169\,\frac{1}{a_0}$$

$\boxed{7-27}$ Write down the wavefunction for the hydrogen atom when the electron's quantum numbers are $n=3, l=2, m_l=-1$.

Using tables 7-1 and 7-2 $\quad \psi_{32-1} = R_{32}\, Y_{2-1}$

$$\psi_{32-1} = \frac{4}{81\sqrt{30a_0^3}} \left(\frac{r}{a_0}\right)^2 e^{-r/3a_0} \cdot \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta\, e^{-i\phi}$$

$$\boxed{\psi_{32-1} = \frac{1}{81} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{1}{a_0^{3/2}} \left(\frac{r}{a_0}\right)^2 e^{-r/3a_0} \sin\theta \cos\theta\, e^{-i\phi}}$$

because $\frac{4}{\sqrt{30}} \cdot \sqrt{\frac{15}{8}} = 1$

<u>EXTRA CREDIT</u>: Check to be sure the wavefunction is normalized.

That is... $\int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \left(\psi^*\psi \cdot r^2 \sin\theta\right) \overset{?}{=} 1$ $\qquad$ Eq. at top of p. 282.

Since $i=\sqrt{-1}$ appears only in the term $e^{-i\phi}$ and $e^{i\phi} e^{-i\phi} = 1$

complex conj.

$$\psi^*\psi = \frac{1}{6561\pi} \cdot \frac{1}{a_0^3} \left(\frac{r}{a_0}\right)^4 e^{-2r/3a_0} \sin^2\theta \cos^2\theta$$

So... $\times r^2 \sin\theta$

$$\int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{1}{6561\pi} \cdot \frac{1}{a_0^7} r^6 e^{-2r/3a_0} \sin^3\theta \cos^2\theta \, d\phi\, d\theta\, dr$$

Can do the ϕ integral trivially... $\times 2\pi$

$$\frac{2}{6561} \frac{1}{a_0^7} \int_0^\infty \int_0^\pi r^6 e^{-2r/3a_0} \sin^3\theta \cos^2\theta \, d\theta\, dr \overset{?}{=} 1$$

Or.. $\frac{2}{6561} \cdot \frac{1}{a_0^7}\left[\int_0^\infty r^6 e^{-2r/3a_0} dr\right]_1 \times \left[\int_0^\pi \sin^3\theta \cos^2\theta \, d\theta\right]_2 \stackrel{?}{=} 1$

Work on these integrals separately...

$\int_0^\infty r^6 e^{-2r/3a_0} dr$ change variables to $\rho = \frac{2r}{3a_0}$ $d\rho = \frac{2}{3a_0} dr$

$$\int_0^\infty \left(\frac{3a_0}{2}\right)^6 \rho^6 e^{-\rho} \left(\frac{3a_0}{2}\right) d\rho = \left(\frac{3a_0}{2}\right)^7 \int_0^\infty \rho^6 e^{-\rho} d\rho$$

Integrate by parts $\int u\,dv = uv - \int v\,du$ let $u = \rho^6 \rightarrow du = 6\rho^5 d\rho$

$v = -e^{-\rho} \leftarrow dv = e^{-\rho} d\rho$

$$\left(\frac{3a_0}{2}\right)^7 \left[\rho^6(-e^{-\rho})\Big|_0^\infty - \int_0^\infty 6\rho^5(-e^{-\rho}) d\rho\right] = \left(\frac{3a_0}{2}\right)^7 \cdot 6 \underbrace{\int_0^\infty \rho^5 e^{-\rho} d\rho}$$

zero at both limits

by analogy, this integral will be, after integration by parts $5\int_0^\infty \rho^4 e^{-\rho} d\rho$

So...

$$\left(\frac{3a_0}{2}\right)^7 \int_0^\infty \rho^6 e^{-\rho} d\rho = \left(\frac{3a_0}{2}\right)^7 \underbrace{6\cdot5\cdot4\cdot3\cdot2\cdot1}_{6!} \underbrace{\int_0^\infty e^{-\rho} d\rho}_{= -e^{-\rho}\big|_0^\infty = 1}$$

$$\left[\int r^6 e^{-2r/3a_0} dr\right]_1 = 6! \left(\frac{3a_0}{2}\right)^7$$

Now the other integral... over θ.

$$\int_0^{\pi} \sin^3\theta \cos^2\theta \, d\theta = \int_0^{\pi} \sin\theta \underbrace{(1-\cos^2\theta)} \cos^2\theta \, d\theta$$

Using trig id. $\sin^2\theta = 1-\cos^2\theta$

$$\int_0^{\pi} (\cos^2\theta - \cos^4\theta) \sin\theta \, d\theta$$

change variables..

let $x = \cos\theta \quad dx = -\sin\theta \, d\theta$

$$\int_1^{-1} (x^2 - x^4)(-dx) = \int_{-1}^{1} (x^2 - x^4) \, dx$$

$$= \left[\frac{1}{3}x^3 - \frac{1}{5}x^5 \right]_{-1}^{1} = \frac{2}{3} - \frac{2}{5}$$

$$= 2\left[\frac{5-3}{15} \right] = \frac{4}{15}$$

So.. $\left[\int_0^{\pi} \sin^3\theta \cos^2\theta \, d\theta = \frac{4}{15} \right]$

Putting it all together...

$$\frac{2}{6561} \cdot \frac{1}{a_0^7} \cdot 6! \left(\frac{3a_0}{2}\right)^7 \cdot \frac{4}{15} \stackrel{?}{=} 1$$

$$\frac{2 \cdot 720 \cdot 3^7}{6561 \cdot 2^7} \cdot \frac{4}{15} \stackrel{?}{=} 1$$

$$\frac{2 \cdot 720 \cdot 2187 \cdot 4}{6561 \cdot 128 \cdot 15} \stackrel{?}{=} 1$$

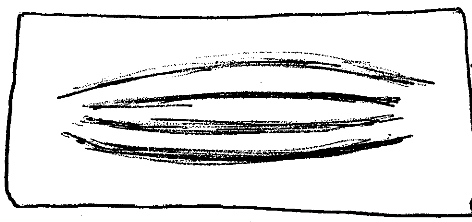
$$1 = 1$$

yes! ✓

$\boxed{7\text{-}33}$ (a) Stern-Gerlach expriment with Yttrium atoms that have $j = 3/2$
How many lines do you expect to see?

You expect to see a line for each possible orientation of the angular momentum (and hence the magnetic dipole moment) vector.

For $j = 3/2$ $\quad m_j = -3/2, -1/2, 1/2, 3/2$ $\quad$ so you should see $\boxed{4}$ lines



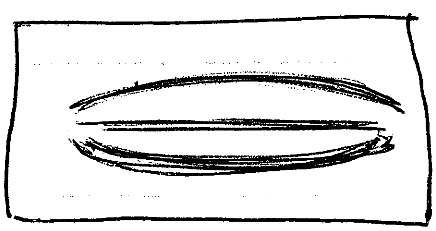
Stern-Gerlach detector screen

(b) If the atoms have $L = 1$ and $S = 0$, then the total angular momentum is $|\vec{J}| = |L - S|, \ldots, L + S$ but this yields only 1 result...

$J = 1 \ldots$

There are $\boxed{3}$ possible orientations for $\vec{J}$ (and hence $\vec{\mu}$) in this case

$m_j = -1, 0, 1$



7-34 The spin-orbit effect in the $n=4$ states of hydrogen.

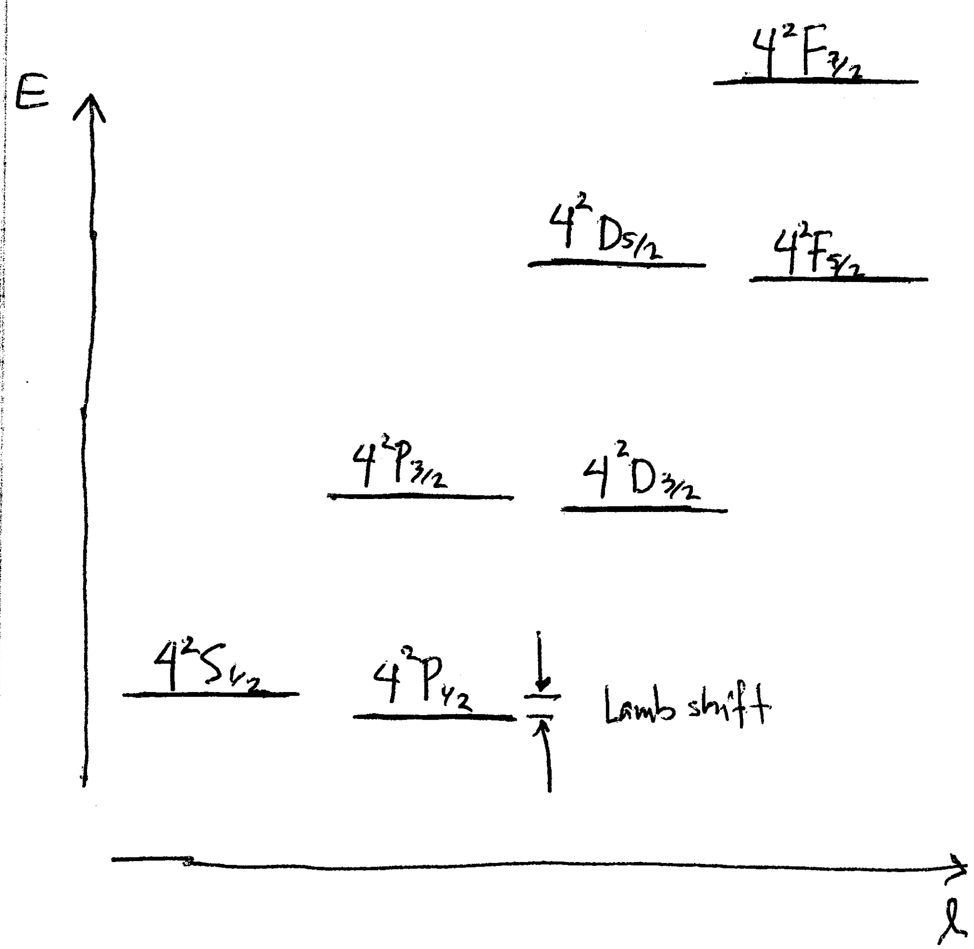
(a) In this level, the orbital angular momentum l can take on values 0, 1, 2, 3 corresponding to spectroscopic labels S, P, D, F.

For $l=0$ $j=\frac{1}{2}$, For $l=1$, $j=\frac{1}{2},\frac{3}{2}$, for $l=2$, $j=\frac{3}{2},\frac{5}{2}$, etc

On page 299, the authors says that "for given n, the energy of the electron increases with increasing l". The spin-orbit effect gives states with larger j a greater energy (see Fig. 7-18). So, the most straightforward answer to this question is that the states, in order of in of increasing energy are ...

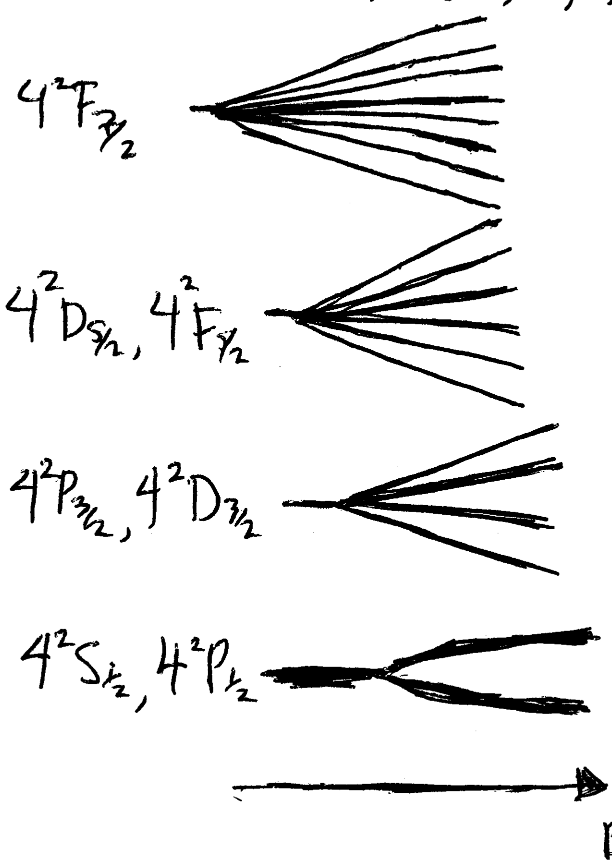
$$\boxed{4^2S_{1/2},\ 4^2P_{1/2},\ 4^2P_{3/2},\ 4^2D_{3/2},\ 4^2D_{5/2},\ 4^2F_{5/2},\ 4^2F_{7/2}}$$

However, see footnote 14, referenced near the top of p. 299 and the preceding sentence. In hydrogen the states with the same j value have nearly identical,, and Lamb shift actually causes a very tiny switch in the order shown above.



(b) If a weak magnetic field is applied, into how many levels will each state split?
The number of levels is equal to the number of m_j values. This is the <u>Zeeman Effect</u>.

If $j = \frac{1}{2}$	$m_j = -\frac{1}{2}, \frac{1}{2}$	2 levels
$= \frac{3}{2}$	$= -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$	4 levels
$= \frac{5}{2}$	$= -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$	6 Levels
$= \frac{7}{2}$	$= -\frac{7}{2}, -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$	8 levels



$\boxed{7-36}$ A hydrogen atom is in the 3D state. (a) What are the possible values of j?

In the D state $l = 2$. Since there is only 1 electron, $S = \frac{1}{2}$

So. $j = 2 - \frac{1}{2} = \boxed{\frac{3}{2}}$ or $j = 2 + \frac{1}{2} = \boxed{\frac{5}{2}}$

$$3^2D_{3/2}, \quad 3^2D_{5/2}$$

(b) What are the possible values for the magnitude of the total angular momentum?

$$|\vec{J}| = \sqrt{j(j+1)}\,\hbar$$

For $j = 3/2$ $\quad |\vec{J}| = \sqrt{\frac{3}{2}\left(\frac{5}{2}\right)}\,\hbar = \boxed{\frac{\sqrt{15}}{2}\hbar}$

For $j = 5/2$ $\quad |\vec{J}| = \sqrt{\frac{5}{2}\left(\frac{7}{2}\right)}\,\hbar = \boxed{\frac{\sqrt{35}}{2}\hbar}$

(c) What are the possible z-components of the total angular momentum?

$L_z = m_j\hbar$ $\quad$ where $m_j = -j, -j+1, \ldots, j-1, j$

For $j = 3/2$ $\quad \boxed{L_z = -\frac{3}{2}\hbar, -\frac{1}{2}\hbar, +\frac{1}{2}\hbar, +\frac{3}{2}\hbar}$

For $j = 5/2$ $\quad \boxed{L_z = -\frac{5}{2}\hbar, -\frac{3}{2}\hbar, -\frac{1}{2}\hbar, +\frac{1}{2}\hbar, +\frac{3}{2}\hbar, +5/2\,\hbar}$

$\boxed{7\text{-}44}$ Write the electron configuration for the ground states of (a) carbon, (b) oxygen, and (c) argon.

(a) Carbon is atomic number $Z=6$, so its electron configuration in the ground state is...

$$\boxed{1s^2 2s^2 2p^2}$$

(b) Oxygen $Z=8$: $\boxed{1s^2 2s^2 2p^4}$

(c) Argon $Z=18$: $\boxed{1s^2 2s^2 2p^6 3s^2 3p^6}$

$\boxed{7\text{-}47}$ Which of the following atoms would you expect to have its ground state split by the spin-orbit interaction: Li, B, Na, Al, K, Ag, Cu, Ga?

In order for there to be spin-orbit interaction in the ground state, there must be non-zero total orbital and spin angular momentum.

Li $1s^2 2s^1$ $\ell = 0$ ... no ground state splitting

* B $1s^2 2s^2 2p^1$ ← net ang. momentum and odd # of electron spins... SPLIT

Na $1s^2 2s^2 2p^6 3s^1$ filled 2p states → no spin or orbital ang. momentum.... $3s^1$ electron has $\ell=0$ no splitting.

* Al $1s^2 2s^2 2p^6 3s^2 3p^1$ ← $\ell=1$ SPLIT

K $1s^2 2s^2 2p^6 3s^2 3p^6 4s^1$ --- no splitting, for same reasons as for Na.

Ag $\underbrace{1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^6 4d^{10}}_{\text{filled sub-orbitals... no net ang. momentum}} 5s^1$ ← $\ell=0$ no splitting

Cu $\underbrace{1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10}}_{\text{filled sub-orbitals....}} 4s^1$ same as Ag no splitting

* Ga $1s^2 2s^2 2p^6 3s^2 3p^6 3d^{10} 4s^2 4p^1$ ← net angular momentum! SPLIT

These three elements have ground state spin-orbit splitting

$\boxed{7-48}$ If the 3s electron in sodium did not penetrate the inner core its energy would be $E_n = -\frac{Z^2 E_o}{n^2}$ with $Z = 1$ due to the shielding of the other 10 electrons and $E_o = 13.6\text{eV}$. $E_3 = -\frac{13.6\text{eV}}{9} = \underline{\underline{-1.51\text{eV}}}$

the same as the $n=3$ level in hydrogen.

But, the measured ionization potential is 5.14V ... not 1.51V, meaning that the 3s electron sees an effective nuclear charge that is greater than 1 .... it penetrates into the inner electron region.

$$-\underset{\substack{\uparrow \\ 5.14\text{V}}}{E_3} = -\frac{Z_{eff}^2 E_o}{3^2} = -Z_{eff}^2 \cdot 1.51\text{V}$$

So... $Z_{eff} = \sqrt{\frac{5.14}{1.51}} = \boxed{1.84}$

$\boxed{7-49}$ What elements have these ground-state electron configurations?

(a) $1s^2 2s^2 2p^6 3s^2 3p^2$ ← add up electrons... $Z = 14$ $\boxed{\text{Si}}$

(b) $1s^2 2s^2 2p^6 3s^2 3p^6 4s^2$ $Z = 20$ $\boxed{\text{Ca}}$