Physics 225 **COMPUTATIONAL MECHANICS** **Winter Term, 2009**

MIGHTY BIG PROBLEM: SOLUTION OF THE LANE-EMDEN EQUATION

BACKGROUND: The Lane-Emden Equation (LEE) is an ordinary, second-order differential equation that has seen applications in physics and astrophysics, particularly in the field of equilibrium configurations of self-gravitating spheres. The $DIFFEQ$ itself is straightforward enough to write down; your challenge is to solve, as throughly as possible, LEE, but also to comment on its physical applications and implications. In fact, the physics behind the usual derivation of LEE is deceptively broad in its reach. As you will discover, I hope, very tiny adjustments to the general equation can be used to investigate a range of phenomena, including (off the top of my head and in my own research experience): white dwarfs, neutrons stars, the proto-earth, Jupiter and Jupiter-like planets, a variety of low- and high-mass stars (including, in composite form, our own Sun), and the collapsing cores of massive stars just prior to supernova explosion. All this, from a delightful equation filled with dimensionless squiggles and two very fundamental assumptions:

1. **Hydrostatic Equilibrium is maintained throughout the star**. This is the condition that the gravitational force downwards in these (1D, spherical) models is precisely balanced by the outward pressure force. Thus, we are ignoring things like rotation, thermonuclear generation, and other outside influences that may complicate the issue. **Question 1: What are these assumptions, what are some other assumptions, and how might they affect the outcome of equilibrium states?**
2. **The spheres are** *polytropes***, namely, they follow a "polytropic equation of state."** An equation of state (EOS) is a mathematical relation between a fluid's state variables– usually pressure (P), mass density (ρ), temperature (T), composition (in most cases related through a mean molecular weight μ), and entropy (S, though sometimes K).

   You are certainly familiar with the ideal gas equation of sate ($IGEOS$):

   $$P = \frac{\rho k T}{\mu m_p},$$

   where k is the Boltzmann constant and m_p is the mass of a proton. Note that for an *isothermal* EOS ($IEOS$):

   $$\frac{P}{\rho} = constant,$$

   under one important assumption (other than T = constant). **Question 2: What is that assumption?** *Barotropes* are fluids with much simpler relationships between P and ρ. The special case of the *Polytropic Equation of State* ($PEOS$) is given by:

   $$P = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}},$$

where K is the *Polytropic Constant*, and is essentially a tracer of entropy (here we are assuming K truly does not vary, in which case the fluid is said to be *isentropic*), γ is the Polytropic Exponent (and often can be related to the more recognizable–or at least more often used–ratio of specific heats from thermal physics), and n is the Polytropic Index. A *Polytrope*, then, is an equilibrium fluid object that obeys $PEOS$ and is usually denoted by the index, n, e.g, "a white dwarf model assumes an $n = 3/2$ polytrope (and by extension, $\gamma = 5/3$).

The relationship between different polytropic indices and states of gas has been an important connection between theory and observation. For example, a gas with polytropic index $n = 3/2$ is a fair approximation to one that is monatomic and isentropic, e.g, an isentropic atomic hydrogen cloud. On the other hand $n = 5/2$ approximates an isentropic diatomic gas (continuing in the similar vein as before, a colder molecular hydrogen cloud). Regarding the structures themselves: as you will see, $n = 0$ corresponds to a uniform density sphere, while $n = 5$ corresponds to an extremely centrally condensed sphere. And so the beauty and utility of LEE and polytropes will become, if not obvious, then at least not entirely too murky.

DERIVATION OF LEE: We will begin with the balance of force somewhere inside a spherical fluid with total mass M, radius R and mass density ρ, in which only pressure and gravity are responsible for acceleration. In that situation, we can write the expression for equilibrium anywhere inside the fluid as:

$$\frac{1}{\rho}\frac{dP}{dr} = -\frac{GM_r}{r^2}, \ (1)$$

where G is the Universal Gravitation constant, and r is our 1D radial coordiate. Note that I have written mass as a function of radius as M_r. An auxilliary equation we must include is the mass conservation equation, namely:

$$\frac{dM_r}{dr} = 4\pi r^2 \rho. \ (2)$$

To proceed, we first multiply Eqn (1) by r^2 and then differentiate both sides with respect to r to get:

$$\frac{d}{dr}(\frac{1}{\rho}r^2\frac{dP}{dr}) = -\frac{d}{dr}(GM_r). \ (3)$$

Next divide both sides by r^2 to get:

$$\frac{1}{r^2}\frac{d}{dr}(\frac{1}{\rho}r^2\frac{dP}{dr}) = -\frac{1}{r^2}\frac{d}{dr}(GM_r). \ (4)$$

Now use use Eqn (2) to eliminate $\frac{dM_r}{dr}$:

$$\frac{1}{r^2}\frac{d}{dr}(\frac{r^2}{\rho}\frac{dP}{dr}) = -4\pi G\rho. \ (5)$$

So far so good, but not necessesarily so clear. Let's, just for fun and because you love to do these things–admit it, you know you do–look at some new variables in order to rewrite Eqn (5) in nondimensional form:

$$\rho = \rho_c \theta^n, \ \ (6)$$

where ρ_c is the central density and θ is dimensionless (note: θ is NOT the polar coordinate angle). Now substitute Eqn (6) into Eqn (5):

$$\frac{1}{r^2}\frac{d}{dr}(\frac{r^2}{\rho_c\theta^n}\frac{dP}{dr}) = -4\pi G\rho_c\theta^n. \ \ (7)$$

At last, bring back $PEOS$, recognizing that $P = K\rho_c^{1+\frac{1}{n}}\theta^{n+1}$. With some rearranging (**Question 3: Show all the steps from Equation 1 to Equation 8 below explicitly.**)

$$[(n+1)\frac{K}{4\pi G}\rho_c^{\frac{1-n}{n}}]\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d\theta}{dr}) = -\theta^n. \ \ (8)$$

Note that we have now collected all constants into the bracketed term in Eqn (8). Since θ is dimensionless, the term in the brackets must have units of length^2, so we further define:

$$r = a\xi, \ \ (9)$$
$$a = [(n+1)\frac{K}{4\pi G}\rho_c^{\frac{1-n}{n}}]^{1/2}, \ \ (10)$$

where ξ is dimensionless. This brings us to the final, glorius form of the Lane-Emden Equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}(\xi^2\frac{d\theta}{d\xi}) = -\theta^n. \ \ (11)$$

Take a moment to bask in its goodness. That's enough. Let's move on.

Boundary Conditions As always, to solve a $DIFFEQ$, you'll need BC's or IC's (if it is a time-dependent equation). Here, we're lucky. At the center, we had better retreive the central density. We also would like to avoid cusps of density–density that approaches infinity–at the center. Thus, given our dimensionless quantities, we have:

$$\theta(\xi = 0) = 1; \ \ (12)$$
$$\frac{d\theta}{d\xi}(\xi = 0) = 0. \ \ (13)$$

Meanwhile, we also have a surface boundary condition, namely: we better hit the surface at the surface. While this sounds a bit trite, what it means is that we can define our radius R in terms of a and ξ. In particular, we'll define $R = a\xi_1$, where ξ_1 represents the value of ξ at which*the last nonzero value of θ occurs.* For a physically realistic system, density (and so θ) must be equal to or grater than zero. For denisty as a decreasing function with radius, ξ_1 marks the position of the surface. Anything beyond it–even if it is *mathematically correct*, is *physically unrealistic.*

Analytic Solutions As you can see from LEE, the nature of the solution will change dramatically with different values of n. Interestingly, there are *only three* known analytic solutions for LEE–for the cases $n = 0$, 1, and 5. They are:

$$\theta_0(\xi) = 1 - \frac{1}{6}\xi^2. \quad (14)$$

$$\theta_1(\xi) = sin(\xi)/\xi. \quad (15)$$

$$\theta_5(\xi) = (1 + \frac{1}{3}\xi^2)^{-1/2}. \quad (16)$$

Properties of Polytropes. Ranging n from a value of 0 to 5 yields a pretty substantial change in physical structure for a polytrope. As you can see for the $n = 0$ case, the result is a uniform density sphere (why? Remember that solving for θ is only part of the problem–you then must solve for ρ using your solution for θ). Each polytrope has a different surface value ξ as well as degree of central concentration–for example, as measured by the central density divided by the average density. You should look up values for the surfaces and the densities for at least $n = 0$, 1, 3/2, 3 and 5 (though be careful about $n = 5$. Why?) and compare them to your lovely computational project below.

NUMERICAL SOLUTION OF THE LANE-EMDEN EQUATION. This is what it's all about, folks. To make the problem more tractable, it is useful to introduce yet another convention. Let's rewrite LEE by expanding the derivative on the left side:

$$\frac{d^2\theta}{d\xi^2} + \frac{2}{\xi}\frac{d\theta}{d\xi} = -\theta^n. \quad (17)$$

Now we can recast this second-order $DIFFEQ$ as two coupled first-order $DIFFEQ$'s (this is something we will do in lecture in other cases as well). Thus, define x:

$$x \equiv \frac{d\theta}{d\xi}. \quad (18)$$

And so

$$\frac{dx}{d\xi} + \frac{2}{\xi}x = -\theta^n. \quad (19)$$

The trick is to update θ, use that to update x, and then use the new x to update θ. And around and around it goes until θ becomes less than zero, in which case you've found the ξ that corresponds to the surface.

You are to solve numerically, verifying all three analytic solutions, and investigating as many nonanalytic cases as you like, LEE. Solutions should include:

1. Solution and plot of θ vs. ξ.
2. Solution and plot of ρ/ρ_c vs. r/R (again, be careful about $n = 5$).
3. Comparison of surface values with known values.
4. Direct comparison with analytic solutions.
5. Answers to all questions contained in this document.
6. A short summary of your work, a discussion of the results, and the possible relevance to physical systems and/or modelling physical systems. The discussion should be word-processed and at least two pages long. Include any pitfalls, discussion of your method (Euler, Runge-Kutta, error, etc.). The more you discuss, the deeper you analyze, the better your score will be.

The assignment is due as indicated on the schedule. There will be no oral presentations. I have cleared the entire last week of the class for you to work on this project, and I will be available for help should you need it.