Mathematics 14C, Fall 00 – Parks.

The Definite Integral and the Fundamental Theorem of Calculus.

Integration is the single most important mathematical technique in all of science. Integrals are used in both computational and theoretical problems to describe and define important quantities and to deduce and predict the behavior of those quantities. You have already seen the indefinite integral of a function: the set of its antiderivatives. We will introduce a number, the *definite integral*, that is computed from the indefinite integral. Then we will introduce circumstances under which the definite integral computes the area of a region in the plane. This will allow us to consider integrals and antiderivatives generally, laying the groundwork for recognizing them in applications.

I. The Definite Integral. For a continuous function $f(x)$, we have defined the *indefinite integral* $\int f(x) \cdot dx$ to be the set of antiderivatives of f, the set of functions $F(x)$ such that $F'(x) = f(x)$. It is a consequence of the Mean Value Theorem that if we can find *one* antiderivative $F(x)$ for $f(x)$, then every other antiderivative differs from F by a constant. This fact is often denoted by writing: $\int f(x) \cdot dx = F(x) + C$ where C is constant.

We are interested in what is called the *definite integral of* $f(x)$. Let $f(x)$ be continuous for every value of x between the two real numbers a and b, then the definite integral is written

$$\int_a^b f(x) \cdot dx$$

The function $f(x)$ is called the *integrand*, the number a is the *lower limit of integration*, and the number b is the *upper limit of integration.* As is the case in the indefinite integral, the differential dx indicates the variable over which $f(x)$ is a function. If $f(x)$ has an indefinite integral, an antiderivative $F(x)$ on $[a, b]$, then

$$\int_a^b f(x) \cdot dx = F(b) - F(a)$$

There are discontinuous functions for which the definite integral is defined. We want to stay close to the main body of applications, which involve continuous functions. We will prove later that every continuous function has an antiderivative, but we will not indulge in a completely thorough discussion of the integral in every possible case.

We defined the definite integral using a specific choice $F(x)$ of antiderivative. What if we had used a different antiderivative? The Mean Value Theorem says that any other antiderivative for $f(x)$ is of the form $F(x) + C$ where C is a constant. Using $F(x) + C$ in place of $F(x)$ in the definition of the definite integral we get

$$\int_a^b f(x) \cdot dx = (F(b) + C) - (F(a) + C) = F(b) + C - F(a) - C = F(b) - F(a)$$

Thus, we compute the same value for the definite integral no matter which antiderivative is used. Another remark which emphasizes the role of antidifferentiation: if $f'(x)$ is continuous between a and b, then we have

$$\int_a^b f'(x) \cdot dx = f(b) - f(a)$$

You have already seen at least one case of this idea: we have considered the motion of an object along, say, the y-axis. We know that $y = \int v \cdot dt$ where v is velocity and t is time. If t goes from time a to time b, the net distance traveled by the car is $y(b) - y(a)$. Notice that

$$y(b) - y(a) = \int_a^b v(t) \cdot dt$$

Thus, when the integrand is velocity, the definite integral over a time interval is the net distance traveled during that time interval. We will see many other interpretations of the definite integral.

Alan Parks, Lawrence University, 1989, revised 1995, 1998, 2000.

Our immediate concern is to get used to the notation for the definite integral. We will do this by touring its elementary properties. First we need just a bit more notation! It is often convenient to write the quantity

$$F(b) - F(a) \qquad \text{as} \qquad F(x)\Big|_a^b$$

You will see the utility of this notation in specific examples, where we write down an antiderivative for $f(x)$ first, and then compute the difference that defines the definite integral.

Second we call attention to a subtlety of the definition. We said that $f(x)$ is continuous *between* a and b. We *did not say* that $a < b$. There are natural situations in which $a > b$; here is a sample computation:

$$\int_2^1 x^2 \cdot dx = \frac{x^3}{3}\Big|_2^1 = \frac{1^3}{3} - \frac{2^3}{3} = -\frac{7}{3}$$

Putting the smaller number as the lower limit, we get

$$\int_1^2 x^2 \cdot dx = \frac{x^3}{3}\Big|_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}$$

It is no accident that the two answers are negatives. In general we have

$$\int_a^b f(x) \cdot dx = -\int_b^a f(x) \cdot dx$$

since, if $F'(x) = f(x)$, then the left side is $F(b) - F(a)$ and the integral on the right side is

$$\int_b^a f(x) \cdot dx = F(x)\Big|_b^a = F(a) - F(b)$$

and this is the negative of the other integral.

It may also have $a = b$. Do you see that the definite integral is 0 in this case?

Next we show that the integral is *linear*, that is, it distributes through sums and over constants. Let $f(x)$ and $g(x)$ be continuous between the two numbers a and b. If $F'(x) = f(x)$ and $G'(x) = g(x)$, then we have

$$(F(x) + G(x))' = F'(x) + G'(x) = f(x) + g(x)$$

Then

$$\begin{aligned}\int_a^b (f(x) + g(x)) \cdot dx &= (F(x) + G(x))\Big|_a^b \\ &= (F(b) + G(b)) - (F(a) + G(a)) \\ &= F(b) - F(a) + G(b) - G(a) \\ &= \int_a^b f(x) \cdot dx + \int_a^b g(x) \cdot dx\end{aligned}$$

For a constant c, if $F'(x) = f(x)$, then $(c \cdot F(x))' = c \cdot F'(x) = c \cdot f(x)$ so that

$$\begin{aligned}\int_a^b c \cdot f(x) \cdot dx &= (c \cdot F(x))\Big|_a^b \\ &= c \cdot F(b) - c \cdot F(a) = c \cdot (F(a) - F(b)) \\ &= c \cdot \int_a^b f(x) \cdot dx\end{aligned}$$

Thus, we see that the definite integral is linear.

Next we want to see what happens when we add integrals with adjoining limits of integration. Suppose that $F'(x) = f(x)$ and compute

$$\begin{aligned}\int_a^b f(x)\cdot dx + \int_b^c f(x)\cdot dx &= F(x)\Big|_a^b + F(x)\Big|_b^c \\ &= (F(b) - F(a)) + (F(c) - F(b)) \\ &= F(b) - F(a) + F(c) - F(b) = F(c) - F(a) \\ &= \int_a^c f(x)\cdot dx\end{aligned}$$

II. Area and Newton's Area Function. What is the definite integral good for? In our remarks at the beginning of these notes, we hinted that the answer is "everything!" The link between the definite integral and applications is subtle, yet there are many ways by which it might be discovered. Each route involves its own set of technical difficulties, and these difficulties seem to require mathematics more advanced than the calculus you are learning. We can avoid the difficulties, no matter which route we choose, if we are willing to take something for granted, without proof, at the outset. It might help to choose an approach involving quantities with which you are familiar, in which you have fairly strong intuition, so that the facts we take for granted are at least believable. We will approach applications through the idea of area in the plane.

You are familiar with the area formulas for triangles, rectangles, circles, and perhaps other shapes. Most of these formulas were discovered by the fourth century B.C. It may come as a surprise that it is not easy to give a rigorous definition of area, nor even to describe the sets of points in the plane that have a well-defined area. Indeed, a complete treatment of area entails mathematics only understood within the last 150 years or so. As we mentioned above, we will avoid technicalities by making explicit assumptions that we will leave unproved.

Let $f(x)$ be continuous and non-negative on $[a, b]$. Then the set of points (x, y) with $a \le x \le b$ and $0 \le y \le f(x)$ defines a region in the plane; we will assume this region has an *area,* which is a non-negative real number. We will denote the area by $\mathbb{A}(f(x), a, b)$, and we will assume the following.

Area Axioms.
a) *(Area of a rectangle.) If $f(x)$ is the constant k, then $\mathbb{A}(f(x), a, b) = k\cdot(b - a)$.*
b) *(Area cut up.) If $a \le c \le b$, then $\mathbb{A}(f(x), a, b) = \mathbb{A}(f(x), a, c) + \mathbb{A}(f(x), c, b)$.*
c) *(Area enclosed in area.) If $g(x)$ is continuous and non-negative on $[a, b]$ and if $g(x) \le f(x)$ there, then $\mathbb{A}(g(x), a, b) \le \mathbb{A}(f(x), a, b)$.*

A simple consequence worth mentioning: if $a = b$, then $\mathbb{A}(f(x), a, b) = 0$. Indeed, $f(x)$ is constant $f(a)$ on the closed interval $[a, a]$ (which consists, after all, of only one point!), and so Axiom (a) applies to show that $\mathbb{A}(f(x), a, a) = f(a)\cdot(a - a) = 0$. This says that the area of a line segment is 0.

Our first insight into computing area is to derive a general "base times height" formula. This fact will be generalized later and called the Integral Mean Value Theorem. The proof uses the Extreme Value Theorem and the Intermediate Value Theorem. You might need to review the statements of these results.

Proposition 1. *Let $a < b$ and suppose that $f(x)$ is continuous and non-negative on $[a, b]$. Then there is a number c in the interval $[a, b]$ such that*

$$\mathbb{A}(f(x), a, b) = f(c)\cdot(b - a)$$

Do you see that $b - a$ is the length of the "base" of the area A. Because $a \le c \le b$, the y-value $f(c)$ is the height above the base of the point $(c, f(c))$. This is why we say that Proposition 1 gives us a base times height formula.

Proof. We are going to trap the area under $y = f(x)$ between two rectangles, one enclosed in the area and one enclosing the area. Draw pictures as you read through this, and notice how the Area Axioms are used. Here is $\mathbb{A}(f(x), a, b)$.

Since $f(x)$ is continuous, the Extreme Value Theorem gives it a minimum. Suppose this minimum occurs when $x = \alpha$, so that $(\alpha, f(\alpha))$ is the lowest point on the graph of $y = f(x)$ for $a \leq x \leq b$. Draw a rectangle with one side along the interval $[a, b]$ and having height $f(\alpha)$. By Axiom (a) the area is $f(\alpha) \cdot (b-a)$. By Axiom (c), the area is less than or equal to the area under $f(x)$. In other words,

$$f(\alpha) \cdot (b - a) \leq \mathbb{A}(f(x), a, b) \tag{1}$$

Now we use the maximum of $f(x)$ to get a large rectangle. The Extreme Value Theorem provides β in $[a, b]$ such that $f(\beta)$ is the maximum of $f(x)$ on the interval. Draw a rectangle with one side along $[a, b]$ and of height $f(\beta)$, and notice that Axiom (a) the area of the rectangle is $f(\beta) \cdot (b - a)$. By Axiom (c), we have

$$\mathbb{A}(f(x), a, b) \leq f(\beta) \cdot (b - a) \tag{2}$$

Putting (1) and (2) together, we get

$$f(\alpha) \cdot (b - a) \leq \mathbb{A}(f(x), a, b) \leq f(\beta) \cdot (b - a)$$

Dividing by the positive number $b - a$ yields

$$f(\alpha) \leq \frac{\mathbb{A}(f(x), a, b)}{b - a} \leq f(\beta)$$

The Intermediate Value Theorem provides a number c in the interval $[a, b]$ such that

$$f(c) = \frac{\mathbb{A}(f(x), a, b)}{b - a}$$

and this is what we wanted. QED

Although Proposition 1 will be useful to us, it is usually difficult, in practice, to find the number c such that $f(c) \cdot (b - a)$ computes the area under $y = f(x)$. We need a better way. We remind you that area formulas have been around for a long time. In the early 1600's, Fermat could see a relationship between the calculation of area and the calculation of tangent lines – in modern terms, between area and the derivative. By the middle of the 1600's many others had seen these same formulas. Newton's teacher, Barrow, suggested to Newton that he try to understand this relationship, and this was one of the problems that led Newton to the discovery of the general calculus.

We want to study a formulation of the idea Newton had for looking at the area under $y = f(x)$. Here is a specific example: $f(x) = x/3$ on $[0, 1]$. Newton suggested introducing a new variable s, and defining a function $A(s)$ that computes the area under $y = x/3$ and over the interval $[0, s]$. In the following picture, $A(s)$ is the area of the shaded part of the graph of $y = x/3$ on $[0, 1]$.

Notice that $A(0) = 0$, being the area of a point, and that $A(1)$ is the total area under $y = x/3$ and above $[0, 1]$. We know the area formula for a triangle, and $A(s)$ is a triangle: of base length s and height $s/3$. Thus

$$A(s) = \frac{1}{2} \cdot s \cdot \frac{s}{3} = \frac{s^2}{6}$$

The key insight is to notice the derivative of this area:

$$\frac{d}{ds}\frac{s^2}{6} = \frac{2s}{6} = \frac{s}{3}$$

The function $s/3$ is the function that forms the top boundary of our region; we had written it as $x/3$ in terms of the variable x, but $s/3$ is the same *function.* In other words, the area function $A(s)$ is an antiderivative of the boundary function $x/3$.

We will look at another example in class. Here is the general case. Let $f(x)$ be continuous with $f(x) \geq 0$ on the closed interval $[a, b]$. Introduce a new variable s and define $A(s)$ to be the area under $y = f(x)$ and above $[a, s]$. The number $A(s)$ is the shaded part of the figure:

Notice that $A(a) = 0$, being the area of a line segment.

Here is the heart of our work: the area function is an antiderivative.

Theorem 2. *Let $f(x)$ be continuous and non-negative on $[a, b]$, and let $A(s)$ be the area under $y = f(x)$ and above the interval $[a, s]$ where $a \leq s \leq b$. Then $A'(s) = f(s)$.*

Proof. To compute $A'(s)$, we need to look at

$$\lim_{\Delta s \to 0} \frac{A(s + \Delta s) - A(s)}{\Delta s}$$

We will look at the ratio on the right in the case when $\Delta s > 0$. The case where $\Delta s < 0$ is similar and will be done in class.

Recall the definition of $A(s)$ and look at the areas represented by $A(s)$ and $A(s + \Delta s)$, and by their difference.

The difference is the area under $y = f(x)$ and above the interval $[s, s + \Delta s]$; in other words $A(s + \Delta s) - A(s) = \mathbb{A}(f(x), s, s + \Delta s)$. We want to apply Proposition 1 to it. There is a number c in the interval $[s, s + \Delta s]$ such that the area under $y = f(x)$ and above $[s, s + \Delta s]$ is the product of $f(c)$ and the length of the interval:

$$f(c) \cdot (s + \Delta s - s) = f(c) \cdot \Delta s$$

Thus,

$$A(s + \Delta s) - A(s) = f(c) \cdot \Delta s \qquad \text{or} \qquad \frac{A(s + \Delta s) - A(s)}{\Delta s} = f(c)$$

The number c depends on s and Δs. As $\Delta s \to 0$, the number c will go to s. Since f is continuous, as $c \to s$, we will have $f(c) \to f(s)$. Thus,

$$f(s) = \lim_{\Delta s \to 0} f(c) = \lim_{\Delta s \to 0} \frac{A(s + \Delta s) - A(s)}{\Delta s} = A'(s)$$

as we wanted. QED

Don't be confused by the use of two different variables: x and s. As we have frequently pointed out, practically any symbol may be used as argument to a given function. Instead of writing the area function $A(s)$, as a function of s, we could equally well write it as a function of the original variable x that came with $f(x)$. Then the conclusion of Theorem 2 is that $A'(x) = f(x)$. In any case, we see that f has an antiderivative.

Theorem 2 also shows us how to compute area under $y = f(x)$ when $f(x) \geq 0$ by evaluating a definite integral! This important application should **not** be taken too far; there will be many other applications of the integral, and the thought "the integral computes area" is dangerously limited. Better to say, "area can be computed using the integral." The difference in tone may seem slight; we will insist on observing the distinction.

Theorem 3. *Let $a < b$ and suppose that $f(x)$ is continuous and non-negative on the closed interval $[a, b]$. Then the area under $y = f(x)$ and above $[a, b]$ is the definite integral of $f(x)$ over $[a, b]$:*

$$\mathbb{A}(f(x), a, b) = \int_a^b f(x) \cdot dx$$

Proof. We have pointed out that $A(a) = 0$ and that $A(b)$ is the area under $y = f(x)$ and above $[a, b]$. Thus, this area is $A(b) = A(b) - A(a)$. Theorem 2 says that $A(x)$ is an antiderivative of $f(x)$. QED

III. Antiderivatives for Continuous Functions. Theorem 2 has as hypothesis that the function $f(x)$ is non-negative; we want to generalize to the setting of an arbitrary continuous function. That such a function has an antiderivative is important, especially since there are continuous functions for which there is no "nice" formula for the antiderivative (e.g. $\cos(x^2)$).

Theorem 4. *Let $f(x)$ be continuous on $[a, b]$. Then there is a function $F(x)$ for which $F'(x) = f(x)$ for all x on $[a, b]$.*

Proof. The proof employs a trick to get us to the hypothesis of Theorem 2. Since $f(x)$ is continuous on $[a, b]$, it has a minimum m there. By the definition of "minimum" we have that $f(x) \geq m$ for all x on $[a, b]$. Thus, $f(x) - m \geq 0$ on $[a, b]$.

The function $f(x) - m$ is continuous and non-negative on $[a, b]$. Theorem 2 says that its area function $A(x)$ is an antiderivative of it. Right now we don't care about area, merely that $A'(x) = f(x) - m$. Put $F(x) = A(x) + mx$ and compute that

$$F'(x) = [A(x) + mx]' = A'(x) + m = (f(x) - m) + m = f(x)$$

so that F is an antiderivative for f. QED

Now we see that the definite integral makes sense whenever $f(x)$ is continuous between the numbers a and b, no matter whether $f(x)$ is positive or negative between the numbers. In the very special case that $f(x) \geq 0$ and $a < b$, we have interpreted the definite integral as an area. What is it in general? In class we will show that the integral can be interpreted as a "net area," but we are more interested in going on to other applications. To get to these applications we need to think about approximating the integral in the way that secant lines approximate the tangent line.

Before we introduce approximations, we need to generalize Proposition 1, which required that $f(x)$ be non-negative. You will need to remember the statement of this result.

Integral Mean Value Theorem. *Let $f(x)$ be continuous on $[a, b]$. Then there is a number c in $[a, b]$ such that*

$$\int_a^b f(x) \cdot dx = f(c) \cdot (b - a)$$

Proof. Theorem 4 gives $f(x)$ an antiderivative $F(x)$. The function $F(x)$ is differentiable on $[a,b]$, and so, by the Mean Value Theorem, there is $c \in [a,b]$ such that $F(b) - F(a) = F'(c)(b-a)$. Since $F'(x) = f(x)$, we see that $F(b) - F(a) = f(c)(b-a)$. On the other hand

$$\int_a^b f(x) \cdot dx = F(b) - F(a) = f(c) \cdot (b-a)$$

QED

IV. Riemann Sums. These sums approximate the definite integral. Because the sums are easy to calculate, one obvious use of them is to provide numerical approximations to an integral. But the real utility of Riemann sums is in allowing us to recognize *abstractly* when a quantity can be calculated using an integral. There is an analogy here with the derivative. We used the slope of a secant line to approximate the value of the derivative. The $\Delta y/\Delta x$ form of the slope of the secant line showed us that the derivative was relevant to many situations, not just tangent lines. Riemann sums will show us that integrals are relevant to many situations, not just the calculation of area. We warn you that until you have worked extensively with such applications, it will be hard to appreciate Riemann sums. In fact, they are difficult to work with. In class, we will start with an application as a way of discovering them. Here, we will give the relevant definitions.

A Riemann sum is determined by the following ingredients (items (b) and (c) will be explained below):

a) A continuous function $f(x)$ on a closed interval $[a,b]$;
b) a sub-division of the interval;
c) a selection in the sub-division.

To get a *sub-division,* we break the closed interval $[a,b]$ into finitely many pieces by choosing x-values intermediate between a and b. Here is how a sub-division might be denoted:

$$a = x_0 < x_1 < x_2 < x_3 < \cdots < x_{n-1} < x_n = b \tag{3}$$

There are many, many ways this can be done. The easiest way is to let the x_j be equally spaced in $[a,b]$. For example, if $[a,b] = [0,1]$, and $n = 7$, then we could have

$$0 = \frac{0}{7} < \frac{1}{7} < \frac{2}{7} < \frac{3}{7} < \frac{4}{7} < \frac{5}{7} < \frac{6}{7} < \frac{7}{7} = 1$$

In general, we do not require the x_j to be equally spaced, they may be chosen arbitrarily.

Choosing the x_j divides up $[a,b]$ into n sub-intervals:

$$[x_0, x_1], \quad [x_1, x_2], \quad \cdots \quad [x_{n-1}, x_n]$$

A *selection* comes about by picking an x-value from each of the sub-intervals: pick c_1 from the first interval $[x_0, x_1]$; pick c_2 from the second interval $[x_1, x_2]$, and so on, picking c_j from $[x_{j-1}, x_j]$ for each $1 \le j \le n$. The selection may be made systematically or completely at random. We can, for example, let c_j be the righthand endpoint of the interval, so that $c_j = x_j$. We could let c_j be the lefthand endpoint; we could let c_j be the midpoint; we can make any choice we wish.

Given the ingredients: $f(x)$ on $[a,b]$, a sub-division as in (3), a selection of c_j, we define the *Riemann sum:*

$$\sum_{j=1}^{n} f(c_j) \cdot (x_j - x_{j-1})$$

When $f(x) \ge 0$, this sum represents a sum of areas of rectangles; we will draw a picture of this in class. This picture will suggest that Riemann sums estimate $\int_a^b f(x)dx$. You will be asked to compute various Riemann sums for several functions over several intervals to get more familiar with this concept.

The following result puts feet to the idea that Riemann sums approximate the integral; this result is used in most calculus texts to motivate applications of the integral.

The *mesh* of a sub-division is the width of its widest sub-interval.

Theorem 5. *Let $f(x)$ be continuous on $[a, b]$. Then a Riemann sum can be made arbitrarily close to the integral of f on $[a, b]$ by making the mesh of the sub-division sufficiently small (for every possible selection).*

We might have time in class to give a proof of Theorem 5 in the following, commonly occurring special case (the proof of the result in general involves advanced calculus).

Theorem 6. *Suppose that there is a positive number M such that $|f'(x)| \leq M$ for all x in $[a, b]$. A Riemann sum using a sub-division of mesh δ will be within $(b-a) \cdot M \cdot \delta$ of $\int_a^b f(x) \cdot dx$.*

The notation for Riemann sums takes some getting used to, but it does help explain the notation for the integral. A Riemann sum looks like a sum of areas of rectangles. Leibnitz imagined the integral itself as a sum of infinitely many, infinitely narrow rectangles: each sub-rectangle of width dx and height $f(x)$ at the point x. The integral notation

$$\int_a^b f(x) \cdot dx$$

is seen as a kind of "super-sum" where, at least intuitively, we add up "all" the rectangles of height $f(x)$ and infinitesimal width dx. We have mentioned that the integral sign $\int$ is an elongated German "s," standing for "sum." We are not trying to define "infinitely narrow rectangles," but to get some idea how Leibnitz conceived the integral and its notation.

V. The Fundamental Theorem.

We summarize what we have done, uniting Theorems 3,4,5.

Fundamental Theorem of Calculus. *Let $f(x)$ be a continuous function on $[a, b]$. Then there is an antiderivative $F(x)$ for $f(x)$ on $[a, b]$ and*

$$\int_a^b f(x) \cdot dx = F(b) - F(a)$$

If $f(x) \geq 0$, then this integral is the area under $y = f(x)$ and above $[a, b]$. The integral can be made arbitrarily close to a Riemann sum for $f(x)$ on $[a, b]$ by making the mesh of the sub-division sufficiently small.

Problems.

Use antiderivatives to compute the following definite integrals:

1. $\int_0^1 (3x-9) \cdot dx$ **2.** $\int_1^{-1} x^5 \cdot dx$ **3.** $\int_2^4 \sqrt{x-1} \cdot dx$ **4.** $\int_0^{\pi} \cos(x) \cdot dx$ **5.** $\int_0^{\pi} \sin(x) \cdot dx$

6. Show that

$$\int_a^b f(x) \cdot dx + \int_b^c f(x) \cdot dx + \int_c^a f(x) \cdot dx = 0$$

(Hint: write $F(x)$ for an antiderivative for $f(x)$ and compute away.)

7. Show that

$$\int_{-a}^{a} \sin(x) \cdot dx = 0$$

8. Draw the curve $y = 2x + 3$ over the interval $[2, 5]$. Use the area formula for a trapezoid to find a formula for $A(s)$, the area under $y = 2x + 3$ and above $[2, s]$. What is $A'(s)$? What is $A'(x)$? Does this agree with Theorem 2?

9. Archimedes found a formula that computes the area bounded by the parabola $y = b - x^2$ (for $b > 0$), the y-axis, and a horizontal line segment at $y = h$ (where $h \leq b$). Let T be the triangle with vertices at $(0, b)$, at $(0, h)$ and at the intersection (to the right of the y-axis) of the parabola and the line $y = h$. Then the area between the parabola and the two lines is 4/3 times the area of T. Use this formula to find the area function $A(s)$ under $y = b - x^2$ and above $[0, s]$ where $0 \leq s \leq \sqrt{b}$. (Hint: the area you want consists of an Archimedes-type area plus the area of a rectangle.)

10. Use Theorem 3 to compute the area under $y = \frac{1}{x^2}$ and above the interval $[1, s]$. What happens to this area as $s \to \infty$?

11. Theorem 4 tells us that $\cos(x^2)$ has an antiderivative $f(x)$. (Don't try to find a formula for f, there is no easy way to write down what it looks like.) Where is $f(x)$ increasing and where is it decreasing for $0 \leq x \leq \sqrt{\pi}$?

12. Theorem 4 says that $1/x$ has an antiderivative $f(x)$ defined for $x > 0$. (The function f will turn out to be a logarithm; we will study this in a couple of weeks. Don't use logarithms on this problem, even if you have already seen them.) Show that $f(2x)$ is an antiderivative for $1/x$, too. So what?

13. Let $f(x) = c$, a constant. Show that every Riemann sum for $f(x)$ on the interval $[a, b]$ is equal to $c \cdot (b - a)$.

14. Compute Riemann sums for the function $f(x) = 1/x$ on the interval $[1, 2]$ using equally spaced subdivisions, 1/10 apart, and (a) the lefthand endpoints as selection points; (b) the righthand endpoints as selection points. Draw a picture that shows that one of these Riemann sums is below the integral and that the other is above the integral. How good an estimate of the integral can you get from these two numbers?

15. In Mathematics 15 you will show that

$$\frac{\pi}{4} = \int_0^1 \frac{1}{1 + x^2} \cdot dx$$

Use Riemann sums to approximate the integral on the right to within 10^{-2}. (Hint: use Theorem 6. Show that the derivative of $1/(1 + x^2)$ is bounded by 2; how small does the mesh have to be?)

Proof of Theorem 6.

We consider a sub-division $a = x_0 < x_1 < \cdots < x_n = b$. Remember that $x_0 = a$ and $x_n = b$ in the sub-division. By what we did in section I, we have

$$\begin{aligned} \int_a^b f(x) \cdot dx &= \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)dx \\ &= \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(x) \cdot dx \end{aligned} \tag{4}$$

Now suppose we have a selection c_j for the sub-division. We consider the difference D between the integral and the Riemann sum, using the summation formula (4) we just derived:

$$\begin{aligned} D = \int_a^b f(x) \cdot dx - \sum_{j=1}^{n} f(c_j) \cdot (x_j - x_{j-1}) \\ &= \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(x) \cdot dx - \sum_{j=1}^{n} f(c_j) \cdot (x_j - x_{j-1}) \\ &= \sum_{j=1}^{n} \left[\int_{x_{j-1}}^{x_j} f(x) \cdot dx - f(c_j) \cdot (x_j - x_{j-1}) \right] \end{aligned} \tag{5}$$

We need to look at each term in (5). Using the Integral Mean Value Theorem, we can write

$$\int_{x_{j-1}}^{x_j} f(x) \cdot dx = f(d_j) \cdot (x_j - x_{j-1}) \quad \text{for some} \quad d_j \quad \text{between} \quad x_{j-1},\, x_j$$

Putting these terms in for the integrals, we find that (5) is

$$\begin{aligned} D &= \sum_{j=1}^{n} [f(d_j) \cdot (x_j - x_{j-1}) - f(c_j) \cdot (x_j - x_{j-1})] \\ &= \sum_{j=1}^{n} (f(d_j) - f(c_j)) \cdot (x_j - x_{j-1}) \end{aligned} \tag{6}$$

We want to estimate the numbers $f(d_j) - f(c_j)$ occurring in (6). To do this, use the Mean Value Theorem to find e_j between d_j and c_j such that

$$f(d_j) - f(c_j) = f'(e_j) \cdot (d_j - c_j)$$

The number $f'(e_j)$ is less than or equal to M, by the hypothesis of this theorem. The numbers d_j and c_j are in the interval $[x_{j-1}, x_j]$, and so their difference $|d_j - c_j|$ is less than or equal to the width of this interval. This width is less than or equal to the mesh δ. Thus, $|d_j - c_j| \leq \delta$, and we see that

$$|f(d_j) - f(c_j)| \leq M \cdot \delta$$

Taking this back to (6), we estimate

$$|D| \leq \sum_{j=1}^{n} M \cdot \delta \cdot (x_j - x_{j-1})$$

The terms $x_j - x_{j-1}$ add up to $b - a$, and so

$$|D| \leq M \cdot \delta \cdot (b - a)$$

and the proof is complete.