The IVT.

This is similar to a handout I give in a first term calculus class. Students are responsible to understand the proof of the theorem and to apply it as indicated.

Before doing this result, we would already have discussed the idea of a *bisection scheme.* Given a closed interval $[a, b]$, a bisection scheme is a method of choosing between the left half or right half of $[a, b]$, then choosing the left or right half of the chosen half, then choosing half of that, and so on.

Intermediate Value Theorem. *Let $f(x)$ be a continuous function on the closed interval $[a, b]$. Suppose that L is a real number between $f(a)$ and $f(b)$. Then there is some c in $[a, b]$ such that $f(c) = L$.*

Proof. Assume first that $f(a) \le L \le f(b)$.

We construct a bisection scheme, starting with $I_0 = [a, b]$. Use the midpoint d of I_0 to divide it into two halves: $[a, d]$ and $[d, b]$. Either $f(d) \le L$ or $f(d) > L$. If $f(d) \le L$, define $I_1 = [d, b]$; if $f(d) > L$, define $I_1 = [a, d]$. In the case $f(d) \le L$, we also have $L \le f(b)$, and so $f(d) \le L \le f(b)$, and so L is between the f-values of the endpoints d and b of I_1. In the other case, that $f(d) > L$, we have $f(a) \le L$, and so $f(a) \le L \le f(d)$, and again L is between the f-values of the endpoints of I_1.

Given that L is between the f-values at the endpoints of I_1, let I_2 be half of I_1 such that L is between the f-values at the endpoints of I_2. Similarly, let I_3 be half of I_2 such that L is between or equal to the values of f at the endpoints of I_3, and get I_4 and I_5 and so on. The Completeness Property gives us a number c in all the I_j. We claim that $f(c) = L$.

Assume that $f(c) < L$. Since $\lim_{x \to c} f(x) = f(c)$, once x is close enough to c, we have $f(x) < L$. Because the intervals I_j squeeze down on c, there is one such interval within this degree of closeness to c. If this interval is $I_j = [\alpha, \beta]$, then the construction of the bisection scheme shows that $L \le f(\beta)$. On the other hand, because β is so close to c, we have $f(\beta) < L$, and this is a contradiction. We conclude that the statement "$f(c) < L$" is false.

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We began the proof by assuming that $f(a) \le L \le f(b)$. The other case is that $f(a) \ge L \ge f(b)$. Multiplying by -1, this is $-f(a) \le -L \le -f(b)$. The function $-f(x)$ is continuous on $[a, b]$ and $-L$ is between $-f(a)$ and $-f(b)$ in the way dealt with by the first part of the proof. Therefore, there is a number c on $[a, b]$ such that $-f(c) = -L$, and this is $f(c) = L$, as needed. QED

Suppose we want to find a solution to the equation $x^5 + x - 1 = 0$. There is no general algebraic formula for the roots of a fifth degree polynomial, and so we might consider numerical approximations instead. Write $f(x) = x^5 + x - 1$, so that $f(x)$ is a continuous function. Notice that $f(0) = -1$ and $f(1) = 1$. Since $f(0) < 0 < f(1)$, the Intermediate Value Theorem tells us that there will be a number c between 0 and 1 such that $f(c) = 0$. In other words, $c^5 + c - 1 = 0$, and c is a solution to the equation!

The proof of the IVT suggests how to look for c. Look at the midpoint $1/2$ of $[0, 1]$. If $f(1/2) < 0$, look for c in $[1/2, 1]$; if $f(1/2) > 0$, look for c in $[0, 1/2]$. Compute that $f(1/2) = -15/32$, so we look in $[1/2, 1]$. The midpoint of this interval is 0.75 and $f(0.75) \approx -0.0127$, so we look for c in $[0.75, 1]$. The midpoint is 0.875 and $f(0.875) \approx 0.3879$, and so we look for c in $[0.75, 0.875]$. We can continue this process as long as we want to narrow down on a solution c. After several steps, we obtain $c \approx 0.75488$.

There are quicker methods of solving for c, but the above illustrates how a proof leads to a practical method of approximation.