DIFFERENTIAL EQUATIONS
MIDTERM SOLUTIONS
FALL 2004

1) For an equation of the form $\frac{dy}{dt} = f(a, y)$, where a is a real parameter, the critical points (equilibrium solutions) usually depend on the value of a. As a steadily increases or decreases, it often happens that at a certain value of a, called a <u>bifurcation point</u>, critical points come together, separate, appear or are lost.

2) An <u>autonomous</u> o.d.e. is one of the form $y' = f(y)$.
If $f(k) = 0$ for some $k \in \mathbb{R}$, the constant function $y(t) = k$ is a solution called an <u>equilibrium solution</u>. Such a solution is called <u>stable</u> if there exists an $\varepsilon > 0$ such that for all $y_0 \in (k-\varepsilon, k+\varepsilon)$, the solution to the IVP

$$\begin{cases} y' = f(y) \\ y(0) = y_0 \end{cases}$$

tends to k as $t \to \infty$.

If this is only true for $y_0 \in (k-\varepsilon, k)$ or for $y_0 \in (k, k+\varepsilon)$ then $y(t) = k$ is called a <u>semistable solution</u>. If no solution to $y' = f(y)$ other than $y(t) = k$ tends to k as $t \to \infty$ then this solution is called <u>unstable</u>.

★ If you have questions about your answer to the essay part of this question, see me★
(which I haven't done here)

3) If f and f_y are continuous in a rectangle

$R = \{|t| \le a, |y| \le b\}$, then there is some interval $|t| \le h \le a$ in which there exists a unique solution to the IVP

$$\begin{cases} y' = f(t, y) \\ y(0) = 0 \end{cases}$$

4) Principle of superposition: Let y_1 and y_2 be solutions to the equation

$$y'' + p(t)y' + q(t)y = 0. \qquad \bigstar$$

Then any linear combination $c_1y_1 + c_2y_2$ is also a solution.

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proof Since y_1 and y_2 are solutions,

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \qquad (1)$$

and

$$y_2'' + p(t)y_2' + q(t)y_2 = 0 \qquad (2)$$

Now plug $c_1y_1 + c_2y_2$ into the LHS of $\bigstar$:

$(c_1y_1 + c_2y_2)'' + p(t)(c_1y_1 + c_2y_2)' + q(t)(c_1y_1 + c_2y_2)$ deriv is linear

$= c_1y_1'' + c_2y_2'' + p(t)(c_1y_1' + c_2y_2') + q(t)(c_1y_1 + c_2y_2)$ distribute

$= c_1y_1'' + c_2y_2'' + p(t)c_1y_1' + p(t)c_2y_2' + q(t)c_1y_1 + q(t)c_2y_2$ rearrange & factor out c_1, c_2

$= c_1\underbrace{(y_1'' + p(t)y_1' + q(t)y_1)}_{(1)} + c_2\underbrace{(y_2'' + p(t)y_2' + q(t)y_2)}_{(2)}$

by (1) (2), this is

$= c_1 \cdot 0 + c_2 \cdot 0 = 0.$

So $c_1y_1 + c_2y_2$ is also a solution of $\bigstar$.

5) Solve $\underline{2t\cos y} + 3t^2y + (t^3 - t^2\sin y - y)y' = 0$

This is of the form

$$M(t,y) + N(t,y)y' = 0$$

where $M(t,y) = 2t\cos y + 3t^2y$

$$N(t,y) = t^3 - t^2\sin y - y$$

So to check if it is exact, we need $M_y = N_t$:

$$\left.\begin{aligned} M_y &= -2t\sin y + 3t^2 \\ N_t &= 3t^2 - 2t\sin y \end{aligned}\right\} \text{ so exact.}$$

To find $\psi(t,y)$ with $\psi_y = N$, $\psi_t = M$:

$$\psi = \int N(t,y)\,dy = \int (t^3 - t^2\sin y - y)\,dy$$

$$= t^3y + t^2\cos y - \frac{y^2}{2} + c(t)$$

Now find its deriv wrt t:

$\psi_t = 3t^2y + 2t\cos y + c'(t)$ this must $= M = 2t\cos y + 3t^2y$

so $c'(t) = 0$ so we can take $c(t) = 0$.

So $\psi(t,y) = t^3y + t^2\cos y - \frac{y^2}{2}$.

The general solution is

$$c = \psi(t,y) = t^3y + t^2\cos y - \frac{y^2}{2}.$$

6) Solve the IVP $\begin{cases} y''-3y'-4y=0 \\ y(1)=0 \\ y'(1)=2 \end{cases}$

First consider characteristic polynomial

$$\lambda^2-3\lambda-4=0$$
$$(\lambda-4)(\lambda+1)=0$$
$$\lambda=4,-1$$

So $c_1e^{4t}+c_2e^{-t}$ is the general solution.

$$0=y(1)=c_1e^4+c_2e^{-1}$$

$$2=y'(1)=4c_1e^4-c_2e^{-1}$$

add:

$$2=5c_1e^4 \quad \Rightarrow \quad c_1=\tfrac{2}{5}e^{-4}$$

plug into 1st eqn

$$0=\tfrac{2}{5}e^{-4}\cdot e^4+c_2e^{-1} \quad \Rightarrow \quad c_2=-\tfrac{2}{5}e$$

So the IVP solution is

$$y=\tfrac{2}{5}e^{4(t-1)}-\tfrac{2}{5}e^{t-1}$$

7) a) Use Euler with $h=.5$ to estimate the solution to the following IVP at $t=2$: $\begin{cases} y'=(1+t)(1+y)=f(t,y) \\ y(0)=2 \end{cases}$

$\begin{cases} y_0=2 \\ t_0=0 \end{cases}$

$\begin{cases} y_1=y_0+h\cdot f(t_0,y_0)=2+.5(1+0)(1+2)=2+.5\cdot 3=3.5 \\ t_1=.5 \end{cases}$

$\begin{cases} y_2=y_1+h\cdot f(t_1,y_1)=3.5+.5(1+.5)(1+3.5)= \\ t_2=1 \end{cases}$

$\begin{cases} y_3=y_2+hf(t_2,y_2)= \\ t_3=1.5 \end{cases}$

$\begin{cases} y_4=y_3+hf(t_3,y_3)= \\ t_4=2 \end{cases}$

So the estimate of $y(2)$ is $y_4=$

7b) Solve the equation and compare the actual value of $y(2)$ with your estimate.

$y' = (1+t)(1+y)$ is separable

$$\int \frac{y'\,dt}{1+y} = \int (1+t)\,dt$$

$$\int \frac{1}{1+y}\,dy = \int (1+t)\,dt$$

$$\ln|1+y| = t + t^2/2 + c$$

$$|1+y| = e^{t+t^2/2+c}$$ let $a = \text{sign}(1+y)e^c$

$$1+y = ae^{t+t^2/2}$$

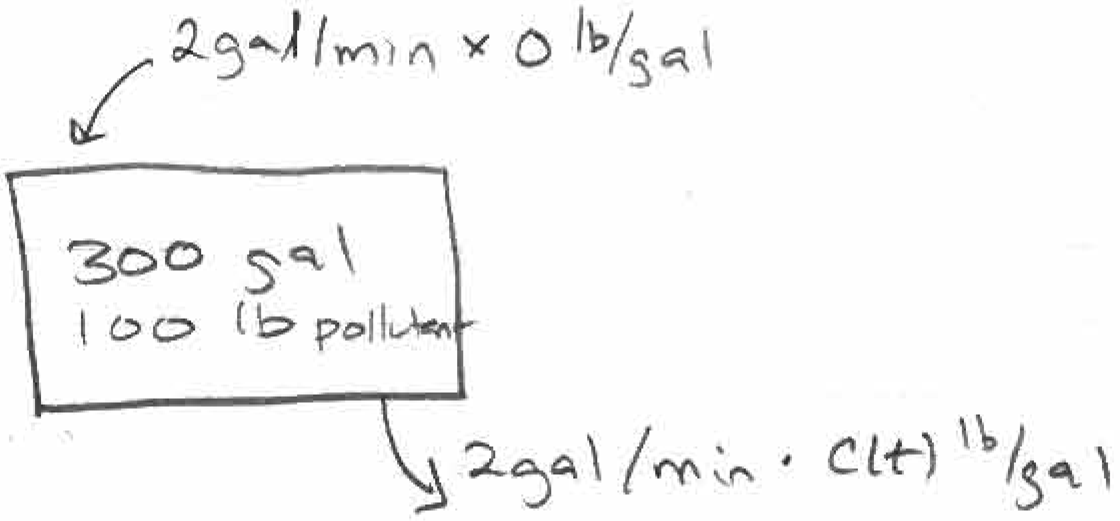
$y = ae^{t+t^2/2} - 1$ is the general solution

$2 = y(0) = ae^0 - 1 = a - 1$ so $a = 3$

Thus $y = 3e^{t+t^2/2} - 1$ is the IVP solution.

$y(2) = 3e^{2+2} - 1 =$

8)



Volume is constant since inflow = outflow

So $\quad C(t) = \dfrac{Q(t)}{300} \qquad Q(0) = 100 \text{ lb}$

$$Q'(t) = 2 \cdot 0 - 2 \cdot C(t) = -2 \cdot \frac{Q(t)}{300} = \frac{-1}{150} Q(t)$$

This is an exponential growth & decay eqn

So $\quad Q(t) = Q(0)\, e^{-t/150} = 100\, e^{-t/150}$

The concentration at t_0 is $\frac{100}{300} = \frac{1}{3}$. So the concentration is $\frac{1}{10}$ of this when $\frac{Q(t)}{300} = \frac{1}{30}$, so when $Q(t) = 10$.

$$10 = Q(t) = 100 e^{-t/150} \quad \Rightarrow \quad e^{-t/150} = \frac{1}{10}$$

$$-t/150 = \ln(1/10)$$

$$t = -150 \ln(1/10)$$

$$\approx$$

9 a) $W(e^{-t^2}, (1+t^2)^{-1}) =$

$$= \begin{vmatrix} e^{-t^2} & (1+t^2)^{-1} \\ (e^{-t^2})' & ((1+t^2)^{-1})' \end{vmatrix}$$

$$= \begin{vmatrix} e^{-t^2} & (1+t^2)^{-1} \\ -2te^{-t^2} & \frac{-2t}{(1+t^2)^2} \end{vmatrix}$$

$$= \frac{-2te^{-t^2}}{(1+t^2)^2} - \frac{-2te^{-t^2}}{(1+t^2)}$$

$$= \frac{-2te^{-t^2}}{(1+t^2)^2} + \frac{2te^{-t^2}(1+t^2)}{(1+t2)^2} = \frac{2t^3e^{-t^2}}{(1+t^2)^2}$$

b) If y_1 and y_2 were two solutions of a 2^{nd} order linear homogeneous equation with $p(t)$ and $q(t)$ continuous on $(-1,1)$ then by Abel's theorem $W(y_1,y_2)$ either $\neq 0$ ever on $(-1,1)$ or $W(y_1,y_2)=0$ everywhere on y_1,y_2.

$W(e^{-t^2}, (1+t^2)^{-1}) = 0$ at $t=0$

$$= \frac{2 \cdot .5^3 \cdot e^{-(.5^2)}}{(1+.5^2)^2} \approx \neq 0$$

at $t = 1/2$

So y_1, y_2 cannot both be solutions of the same such equation.