Midterm Exam
Solutions
Fall 2005

1. An equation of the form $M(t,y) + N(t,y)y' = 0$ is <u>exact</u>, i.e. there is a function $\psi(t,y)$ with $\psi_t = M$ and $\psi_y = N$, iff $M_y = N_t$.

2. An autonomous family of equations $y' = f(a,y)$ depending on a parameter value a may have values a_0 of that parameter, called <u>bifurcation points</u> where the number or stability of equilibrium solution is different for equations with $a < a_0$ than for $a > a_0$

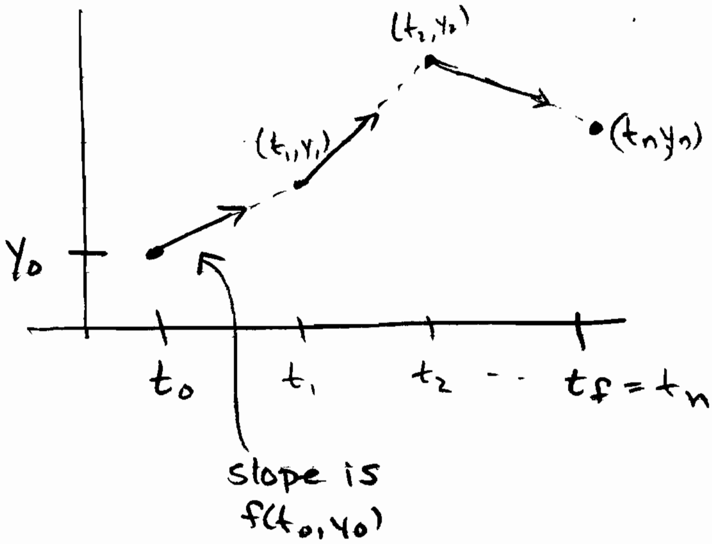
3 a) <u>Principle of Induction</u>
let $A \subset \mathbb{N}$. If A satisfies
1) $1 \in A$
2) if $k \in A$ then $k+1 \in A$
then $A = \mathbb{N}$.

b) <u>Claim</u> $\frac{d^n}{dx^n}(x^n) = n!$
<u>proof (by induction)</u>
base case: If $n = 1$, then $\frac{d}{dx}(x) = 1 = 1!$ ✓
inductive hypothesis: Assume $\frac{d^k}{dx^k}(x^k) = k!$
then we need $\frac{d^{(k+1)}}{dx^{(k+1)}}(x^{k+1}) = (k+1)!$
But
$$\frac{d^{(k+1)}}{dx^{(k+1)}}(x^{k+1}) = \frac{d^k}{dx^k}\left[\frac{d}{dx}(x^{k+1})\right]$$
$$= \frac{d^k}{dx^k}\left[(k+1)x^k\right] = (k+1)\frac{d^k}{dx^k}(x^k) = (k+1)\cdot k!$$ by inductive hypothesis
$= (k+1)!$ ✓
Thus it is true for all n.

4. Euler's Method is a way to approximate the value of $y(t_f)$ where y is the solution to a 1^{st} order IVP $\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$

and where $t_f > t_0$.

It works by using the slope field diagram to approximate the graph of y:



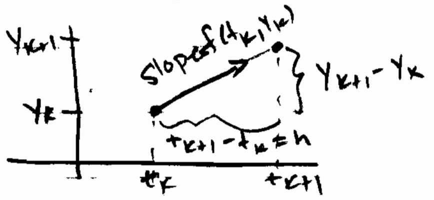
Start by subdividing the interval (t_0, t_f) into n equal parts of "step length" $h = \frac{t_f - t_0}{n}$.

For each step, start where the last step left you (at (t_k, y_k)) and use the slope field arrow to point you to a new point over t_{k+1}. Use the final y value, y_n, over $t_f = t_n$ as your approximation of $y(t_f)$.

The values of t_k are $t_k = t_0 + kh$.

The values of y_k are defined recursively:

$$y_{k+1} = y_k + hf(t_k, y_k) \quad \text{since}$$



$f(t_k, y_k) = \text{slope} = \frac{y_{k+1} - y_k}{h}$

so $y_{k+1} = y_k + hf(t_k, y_k)$.

5. Existence & Uniqueness Thm for 1^{st} order ODE's:
Let f and f_y be continuous on a rectange $R = \{\alpha < t < \beta, \gamma < y < \delta\}$ containing the point (t_0, y_0). Then there exists an $h > 0$ such that there exists a solution $\phi(t)$ to the IVP

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$ on the interval $t_0 - h < t < t_0 + h$

and on this interval, this solution is unique.

6. Thm 3.2.3 Let y_1 and y_2 be two solutions to the equation $y'' + p(t)y' + q(t)y = 0$. If $W(y_1, y_2)(t_0) = y_1'(t_0)y_2(t_0) - y_1(t_0)y_2'(t_0) \neq 0$ for some $t_0 \in I$ then any IVP

$$\begin{cases} y'' + p(t)y' + q(t)y = 0 \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$$

has a solution of the form $c_1 y_1 + c_2 y_2$.

Thm 3.2.4 Let p and q be continuous on I, y_1 and y_2 be solutions to $y'' + p(t)y' + q(t)y = 0$, $(*)$
If $W(y_1, y_2)(t_0) \neq 0$ for some $t_0 \in I$ then every solution to $(*)$ is of the form $c_1 y_1 + c_2 y_2$.

pf By the principle of superposition, any linear combination $c_1 y_1 + c_2 y_2$ is a solution to $(*)$. So in thm. 3.2.3, we only need to show that we can find one that solves the initial conditions.

Since $(c_1y_1+c_2y_2)' = c_1y_1' + c_2y_2'$,

this is true if we can find c_1, c_2 with

$$c_1y_1(t_0) + c_2y_2(t_0) = y_0$$
$$c_1y_1'(t_0) + c_2y_2'(t_0) = y_0'$$

Rewrite this as a matrix equation:

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$$

This has a solution for all $\begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$ if and only if the matrix is invertible, which is if and only if its determinant $\neq 0$: ie, $y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0$. This is exactly the condition that $W(y_1, y_2)(t_0) \neq 0$.

To prove theorem 3.2.4 from here, we note that any solution $\phi(t)$ of $(*)$ on I satisfies the initial value problem it defines for $(*)$ with initial values $\begin{cases} y(t_0) = y_0 = \phi(t_0) \\ y'(t_0) = y_0' = \phi'(t_0) \end{cases}$.

And, by uniqueness, it is the unique solution to this IVP on I.

However, by 3.2.3, we know we can choose c_1, c_2 so that $c_1y_1 + c_2y_2$ is a solution to that IVP. Hence, we must have $\phi = c_1y_1 + c_2y_2$. Since we chose ϕ arbitrarily, this shows any solution to $(*)$ must equal some function of the form $c_1y_1 + c_2y_2$. ∎

7. a) $100{,}000 - y$ = # people who have not heard rumor at time t.

b) $y' = \lambda y(100{,}000 - y)$

c) $y' = -\lambda y(y - 100{,}000)$

$$\frac{y'}{y(y-100{,}000)} = -\lambda$$

$$\frac{A}{y} + \frac{B}{y-100{,}000} = \frac{1}{y(y-100{,}000)}$$

$$A(y-100{,}000) + By = 1$$

let $y = 0$

$$A = \frac{-1}{100{,}000}$$

let $y = 100{,}000$

$$B = \frac{1}{100{,}000}$$

2 So $\frac{1}{100{,}000}\left(\frac{-1}{y} + \frac{1}{y-100{,}000}\right) = \frac{1}{y(y-100{,}000)}$

2 $\int \frac{1}{100{,}000}\left(\frac{-1}{y} + \frac{1}{y-100{,}000}\right) dy = \int -\lambda \, dt$

2 $\frac{1}{100{,}000}\left[-\ln y + \ln(y-100{,}000)\right] = -\lambda t + c$

$$\ln \frac{y-100{,}000}{y} = -100{,}000\lambda t + c$$

$$\frac{y-100{,}000}{y} = ke^{-100{,}000\lambda t}$$

$$y - 100{,}000 = yke^{-100{,}000\lambda t}$$

2 $y - yke^{-100{,}000\lambda t} = 100{,}000$

1 $y(0) = 1$ means

$$1 = \frac{100{,}000}{1-k} \Rightarrow 1-k = 100{,}000$$
$$k = -99{,}999$$

1 So $y = \dfrac{100{,}000}{1+99{,}999\, e^{-100{,}000\lambda t}}$

1 Also, $y(7) = 10{,}000$

$$10{,}000 = \frac{100{,}000}{1+99{,}999\, e^{-700{,}000\lambda}}$$

$$\Rightarrow 1+99{,}999\, e^{-700{,}000\lambda} = 10$$
$$99{,}999\, e^{-700{,}000\lambda} = 9$$

$$e^{-700{,}000\lambda} = \frac{9}{99{,}999} = \frac{1}{11{,}111}$$

$$-700{,}000\lambda = \ln \frac{1}{11{,}111}$$

$$\lambda = -\frac{1}{700{,}000} \ln \frac{1}{11{,}111}$$
$$= .000013$$

1 So $\boxed{y = \dfrac{100{,}000}{1+99{,}999\, e^{-1.33t}}}$

d) $50{,}000 = \dfrac{100{,}000}{1+99{,}999\, e^{-1.33t}} \Rightarrow 1+99{,}999\, e^{-1.33t} = 2$

$$99{,}999\, e^{-1.33t} = 1$$

$$e^{-1.33t} = \frac{1}{99{,}999}$$

$$t = \frac{-1}{1.33} \ln\left(\frac{1}{99{,}999}\right)$$

$$\boxed{t = 8.65633 \text{ days}}$$

8. $y' = \left((1+x^2)^{-1}\right)' = -1(1+x^2)^{-2} \cdot 2x$

$$y' + 2xy^2 = \frac{-2x}{(1+x^2)^2} + 2x \cdot \left(\frac{1}{1+x^2}\right) = 0 \quad \checkmark$$

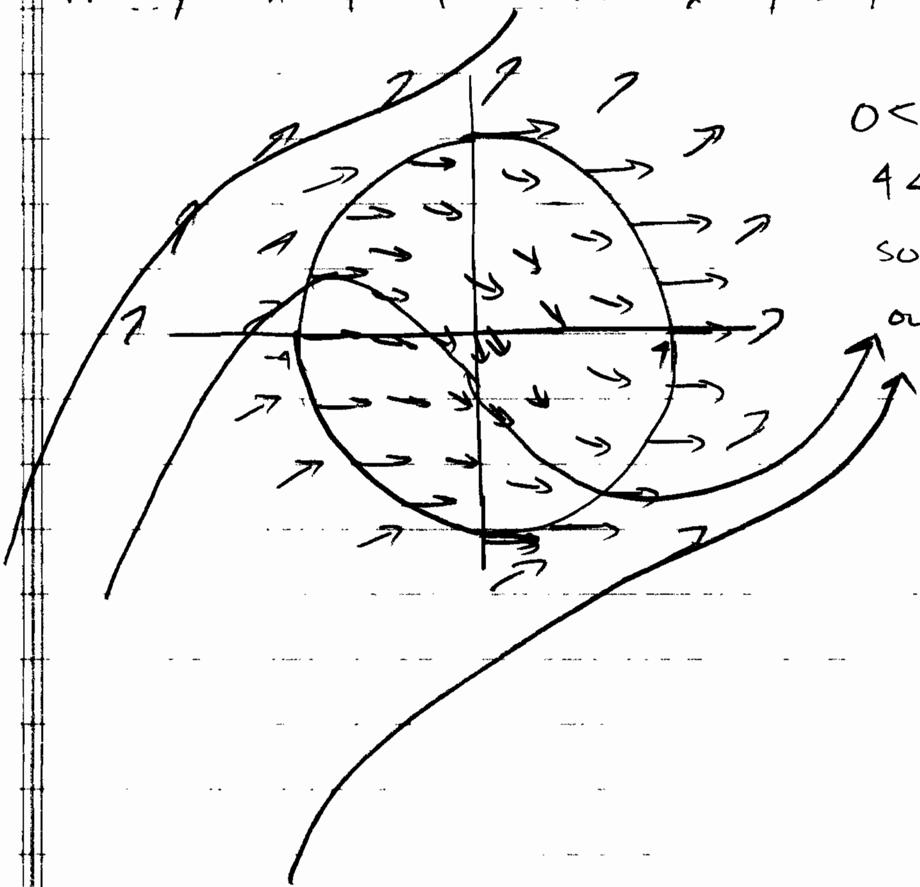
9. $y' = x^2 + y^2 - 4 = 0$ if $x^2 + y^2 = 4$

$0 < y' = x^2 + y^2 - 4$ if

$4 < x^2 + y^2$

so inside $y' < 0$

outside $y' > 0$



10. a) Characteristic eqn: $\lambda^2 - \lambda - 6 = 0$

$(\lambda - 3)(\lambda + 2)$

$y_1 = e^{3t}$, $y_2 = e^{-2t}$ are solutions

$y_1' = 3e^{3t}$ $\quad y_2' = -2e^{-2t}$

$W(y_1, y_2)(t) = -2e^{3t}e^{-2t} - 3e^{3t}e^{-2t}$

$= -5e^{t} \neq 0$ ever so it is a fundamental set.

So the general solution is $c_1 e^{3t} + c_2 e^{-2t}$.

b) $$\begin{pmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \frac{1}{-2-3} \begin{pmatrix} -2 & -1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$= \frac{-1}{5} \begin{pmatrix} -3 \\ -7 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 7/5 \end{pmatrix}$ so $\frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}$ is the solution.

Check: $y(0) = \frac{3}{5}e^{3\cdot 0} + \frac{7}{5}e^{-2\cdot 0} = \frac{3+7}{5} = \frac{10}{5} = 2$ ✓

$y'(0) = \frac{3}{5} \cdot 3e^{3\cdot 0} - \frac{7}{5} \cdot 2e^{-2\cdot 0} = \frac{9}{5} - \frac{14}{5} = -\frac{5}{5} = -1$ ✓

c) $W(e^{rt}, e^{st}) = \begin{vmatrix} e^{rt} & e^{st} \\ re^{rt} & se^{st} \end{vmatrix} = e^{rt} \cdot se^{st} - re^{rt}e^{st}$

$= (s-r)e^{(r+s)t} \neq 0$ ever if $s \neq r$.

11. $y' - y = 11/8\, e^{-x/3}$ is linear 3

$\mu' = -\mu \Rightarrow \mu = e^{-x}$ 3

$(e^{-x}y)' = 11/8\, e^{-x} e^{-x/3}$

$e^{-x}y = \int 11/8\, e^{-4x/3}\, dx$ 3

$e^{-x}y = 11/8 \cdot \frac{-3}{4} e^{-4x/3} + c$ 3

$y = \frac{-33}{32} e^{-x/3} + ce^{x}$ 3