D.E. HW #5 selected Solutions

4.4: 11) Assume $S = \{v_1, \dots, v_n\}$ is a linearly independent set of vectors. Then we claim any subset of S with at least one vector is linearly independent, too.

(by contradiction: Assume not and show hypothesis is contradicted)

pf Assume this is not true and that there is some subset of S that is linearly dependent. By rearranging the elements of S, we can assume without loss of generality (WLOG) that this subset is the 1st r vectors $\{v_1, \dots, v_r\}$.

So we assume there are $k_1, \dots, k_r$ not all 0 so that $k_1 v_1 + \dots + k_r v_r = 0$.

But then $k_1 v_1 + \cdots + k_r v_r + 0 v_{r+1} + \cdots + 0 v_n = 0$ and this is a linear combination of all the vectors in S where not all coefficients $= 0$. But this contradicts our assumption that S was linearly independent. So it is not possible to find a linearly dependent subset of S. ∎

4.4.16) Prove that a set with two or more vectors is linearly dependent if and only if at least one of the vectors may be expressed as a linear combination of the remaining others

proof We are asked to show a statement of the form $A \Longleftrightarrow B$, ie $A \Longrightarrow B$ and $B \Longrightarrow A$.
So there are two steps:
lin dep $\Rightarrow$ lin comb: Assume A is linearly dependent. Then there exist $k_1, \ldots, k_n$ not all 0 such that $k_1 v_1 + \cdots + k_n v_n = 0$. By reordering, we may assume that the nonzero coefficient is k_1. Then

$k_1 v_1 = -k_2 v_2 - k_3 v_3 - \cdots - k_n v_n$ and $k_1 \neq 0$ so

$v_1 = \frac{-k_2}{k_1} v_2 - \frac{k_3}{k_1} v_3 - \cdots - \frac{k_n}{k_1} v_n$, which is a linear combination of $v_2, \ldots, v_n$ ✓

lin comb $\Rightarrow$ lin dep: Again by relablelling if necessary we may assume v_1 is the vector that is a linear combination of the others:

$$v_1 = c_2 v_2 + \cdots + c_n v_n.$$

Thus $-v_1 + c_2 v_2 + \cdots + c_n v_n = 0$ and this is a linear combination of $v_1, \ldots, v_n$ where the coefficient of v_1 is $-1 \neq 0$. Thus the set is linearly dependent ✓