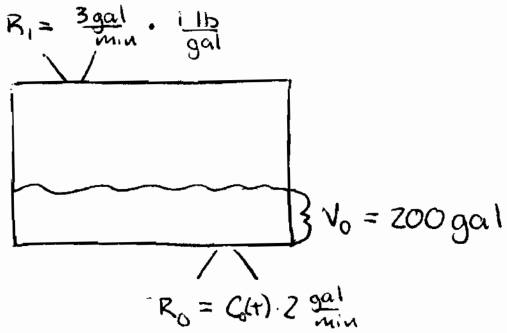
D.E. Hw #2 Selected Solutions

Section 2.3: #4



$R_I = 3\,lb/min$

$R_o = C_o(t) \cdot 2\,gal/min$

$V(t) = 200\,gal + t$

$Q(t) =$ amount of salt

$$C(t) = Q(t)\,/\,V(t)$$

$$\frac{dQ}{dt} = R_I \cdot C_I(t) - R_o \cdot C_o(t)$$

$$\frac{dQ}{dt} = 3 - 2Q(t)/V(t)$$

$$= 3 - 2\left(Q(t)/(200+t)\right)$$

Use the method of integrating factors to solve.

$$Q(t) = 200 + t + \frac{C}{(200+t)^2}$$

$$Q(0) = 100\text{ lbs} \Rightarrow 100 = 200 + C/200^2$$

$$C = -100(200)^2$$

Thus, $Q(t) = 200 + t - \dfrac{100(200)^2}{(200+t)^2}$

The solution begins to overflow when $V = 500$ gal, or at $t = 300$ min.

$$Q(300) = 200 + 300 - 100(200)^2/(300+200)^2$$

$$= 484 \text{ lbs of salt}$$

Thus, $C(300) = \dfrac{484}{500}\dfrac{lbs}{gal} = \dfrac{121}{125}\dfrac{lbs}{gal}$

As $t \to \infty$, $\lim_{t\to\infty} C_T = \lim_{t\to\infty} 1 - \dfrac{100(200)^2}{(t+200)^2} = 1\,lb/gal$

Thus 1 lb/gal is the limiting concentration

2.3. 23(a). Measure the positive direction of motion *downward*. Based on Newton's 2*nd* law,
the equation of motion is given by

$$m\frac{dv}{dt} = \begin{cases} -0.75\,v + mg &, \ 0 < t < 10 \\ -12\,v + mg &, \ t > 10 \end{cases}.$$

Note that gravity acts in the *positive* direction, and the drag force is *resistive*. During the first ten seconds of fall, the initial value problem is $dv/dt = -v/7.5 + 32$, with initial velocity $v(0) = 0\,fps$. This differential equation is separable and linear, with solution $v(t) = 240(1 - e^{-t/7.5})$. Hence $v(10) = 176.7\,fps$.

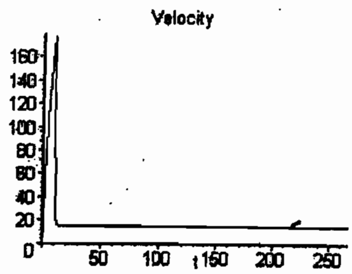
(b). Integrating the velocity, with $x(t) = 0$, the distance fallen is given by

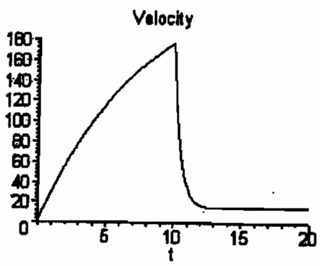
$$x(t) = 240\,t + 1800\,e^{-t/7.5} - 1800.$$

Hence $x(10) = 1074.5\,ft$.

(c). For computational purposes, reset time to $t = 0$. For the remainder of the motion, the initial value problem is $dv/dt = -32v/15 + 32$, with specified initial velocity $v(0) = 176.7\,fps$. The solution is given by $v(t) = 15 + 161.7\,e^{-32t/15}$. As $t \to \infty$, $v(t) \to v_L = 15\,fps$. Integrating the velocity, with $x(0) = 1074.5$, the distance fallen after the parachute is open is given by $x(t) = 15\,t - 75.8\,e^{-32t/15} + 1150.3$. To find the duration of the second part of the motion, estimate the root of the transcendental equation $15\,T - 75.8\,e^{-32T/15} + 1150.3 = 5000$. The result is $T = 256.6$ *sec*.

(d).





2.5. 22(*a*). The equilibrium points are at $y^* = 0$ and $y^* = 1$. Since $f'(y) = \alpha - 2\alpha y$, the equilibrium solution $\phi = 0$ is *unstable* and the equilibrium solution $\phi = 1$ is *asymptotically stable.*

(*b*). The ODE is separable, with $[y(1-y)]^{-1}dy = \alpha\, dt$. Integrating both sides and invoking the initial condition, the solution is

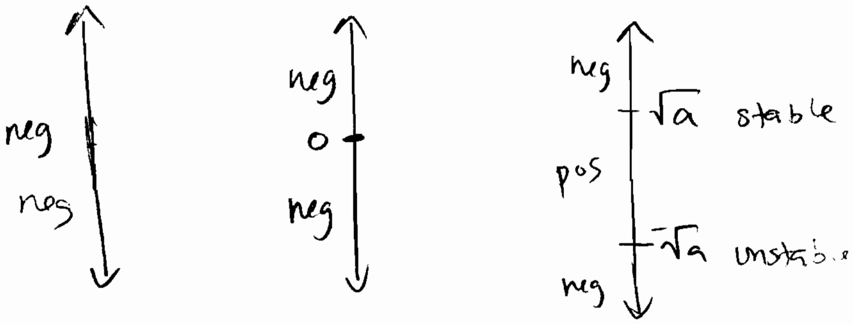
$$y(t) = \frac{y_0\, e^{\alpha t}}{1 - y_0 + y_0\, e^{\alpha t}}\,.$$

It is evident that (independent of y_0) $\lim_{t\to-\infty} y(t) = 0$ and $\lim_{t\to\infty} y(t) = 1$.

2.5.25 $\frac{dy}{dt} = a - y^2$

a) $y = \pm\sqrt{a}$ are critical points if $a > 0$
$y = 0$ is the single critical pt if $a = 0$
none if $a < 0$

b)

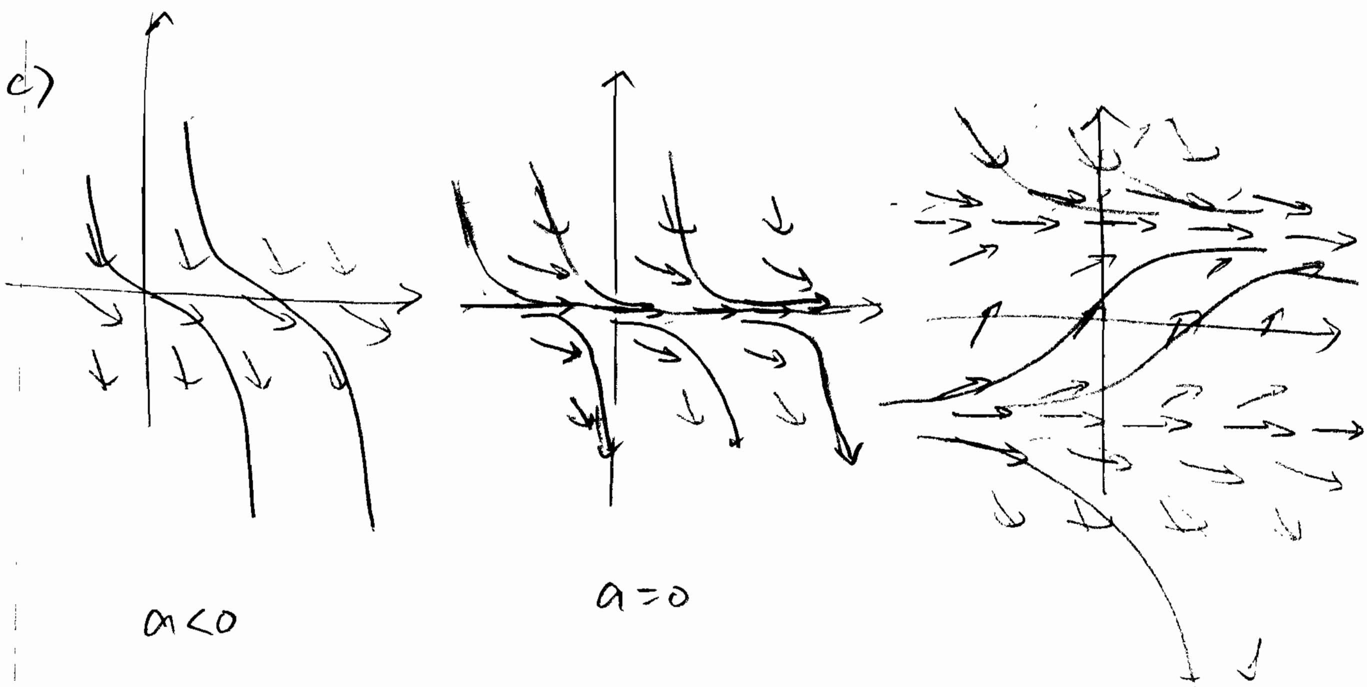


$a < 0$ $a = 0$ $a > 0$

semistable

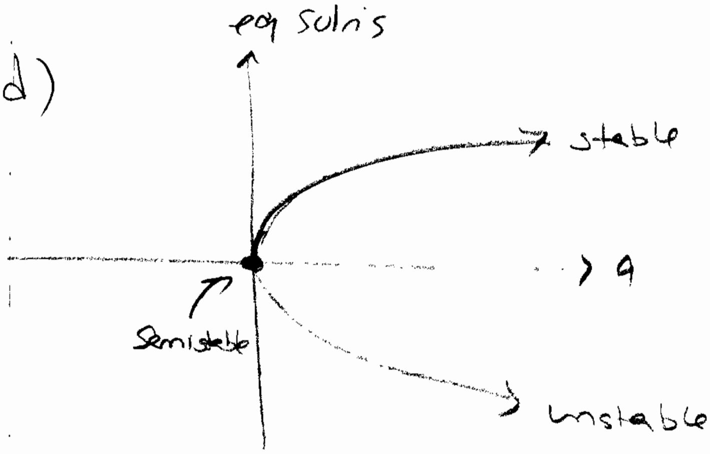
one stable, one unstable

c)



$a < 0$ $a = 0$

d)



2.6.17

$$\psi_y = N(x,y)$$

so $\quad \psi = \int_{y_0}^{y} N(x,t)\,dt + h(x)$

and $\quad \psi_x = \int_{y_0}^{y} N_x(x,t)\,dt + h'(x) = M(x,y)$

so

$$h'(x) = M(x,y) - \int_{y_0}^{y} N_x(x,t)\,dt$$

$$= M(x,y) - \int_{y_0}^{y} M_y(x,t)\,dt$$

$$= M(x,y) - \left(M(x,y) - M(x,y_0)\right)$$

$$= M(x,y_0)$$

So $\quad h(x) = \int_{x_0}^{x} M(s,y_0)\,ds$

eg take

and $\quad \psi = \int_{x_0}^{x} M(s,y_0)\,ds + \int_{y_0}^{y} N(x,t)\,dt$

2.6.17 - alternative solution

$$\Psi(x,y) = \int_{x_0}^{x} M(s,y_0)\,ds + \int_{y_0}^{y} N(x,t)\,dt$$

Need $\Psi_x = M$, $\Psi_y = N$.

$$\Psi_x(x,y) = M(x,y_0) + \int_{y_0}^{y} N_x(x,t)\,dt$$

By FTC & using continuity to bring $\frac{\partial}{\partial x}$ in $\int$.

$$= M(x,y_0) + \int_{y_0}^{y} M_y(x,t)\,dt$$

since $M_y = N_x$

$$= M(x,y_0) + M(x,y) - M(x,y_0) = M(x,y) \checkmark$$

FTC

$$\Psi_y(x,y) = \frac{\partial}{\partial y}\int_{x_0}^{x} M(s,y_0) + N(x,y)$$

by FTC

$$= 0 + N(x,y)$$

Since $M(s,y_0)$ is <u>constant</u> in y (y_0 is fixed!)

$$= N(x,y) \checkmark$$