D.E: Selected Solutions to Hw #1

Section 1.2: 7

(a) The general solution is $p(t) = 900 + ce^{t/2}$, that is, $p(t) = 900 + (p_0 - 900)e^{t/2}$. With $p_0 = 850$, the specific solution becomes $p(t) = 900 - 50e^{t/2}$. This solution is a decreasing exponential, and hence the time of extinction is equal to the number of months it takes, say t_f, for the population to reach zero. Solving $900 - 50e^{t_f/2} = 0$, we find that $t_f = 2\ln(900/50) = 5.78$ months.

(b). The solution $p(t) = 900 + (p_0 - 900)e^{t/2}$, is a decreasing exponential as long as $p_0 < 900$. Hence $900 + (p_0 - 900)e^{t_f/2} = 0$ has only one root, given by

$$t_f = 2\ln\left(\frac{900}{900 - p_0}\right).$$

(c) The answer in part (b) is a general equation relating time of extinction to the value of the initial population. Setting $t_f = 12$ months, the equation may be written as

$$\frac{900}{900 - p_0} = e^6,$$

which has solution $p_0 = 897.7691$. Since p_0 is the initial population, the appropriate answer is $p_0 = 898$ mice.

Section 1.3: 30

Another way to derive the pendulum equation is based on the principle of conservation of energy

(a) Show that the kinetic energy T of the pendulum in motion is

$$T = \frac{1}{2} mL^2 \left(\frac{d\theta}{dt}\right)^2.$$

The kinetic energy of a particle of mass m is given by $T = \frac{1}{2}mv^2$, in which v is its speed, A particle in motion on a circle of radius L has speed $L\left(\frac{d\theta}{dt}\right)$, where θ is its angular position and $d\theta/dt$ is its angular speed.

(b) Show that the potential energy V of the pendulum, relative to its rest position is $V = mgL(1-\cos\theta)$.
Gravitational potential energy is given by $V = mgh$, where h is the height above a certain datum. Choosing the lowest point of the swing as the datum ($V=0$), it follows from trigonometry that $h = 1-\cos\theta$.

(c) From parts (a) and (b),

$$E = \frac{1}{2} mL^2 \left(\frac{d\theta}{dt}\right)^2 + mgL(1-\cos\theta)$$

Applying the Chain Rule for Differentiation,

$$\frac{dE}{dt} = mL^2 \frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + mgL\sin\theta\frac{d\theta}{dt}$$

Setting $dE/dt = 0$ and dividing both sides of the equation by $d\theta/dt$ results in

$$mL^2\frac{d^2\theta}{dt^2} + mL\sin\theta = 0$$

which leads to Equation (12).

Section 2.1: 30

Find the value of y_0 for which the solution of the initial value problem

$$y' - y = 1 + 3\sin t, \quad y(0) = y_0$$

remains finite as $t \to \infty$.

Let $\mu(t) = e^{-t}$:

$$y' - y = 1 + 3\sin t$$
$$e^{-t}y' - e^{-t}y = e^{-t} + 3e^{-t}\sin t$$
$$\int (e^{-t}y)'\,dt = \int (e^{-t} + 3e^{-t}\sin t)\,dt$$
$$e^{-t}y = -e^{-t} + -\tfrac{3}{2}e^{-t}(\cos t + \sin t) + c$$
$$y = -1 - \tfrac{3}{2}(\cos t + \sin t) + Ce^{t}$$

Now apply the initial condition:

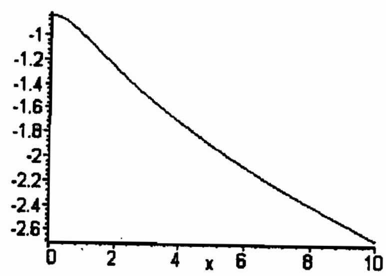
$$y(0) = -1 - \tfrac{3}{2}(\cos 0 + \sin 0) + Ce^{0} = y_0$$
$$y_0 = C - \tfrac{5}{2}$$
$$C = y_0 + \tfrac{5}{2}$$
$$y = -1 - \tfrac{3}{2}(\cos t + \sin t) + (y_0 + \tfrac{5}{2})e^{t}$$

As $t \to \infty$, the term containing e^{t} will dominate the solution. In order for the solution of the initial value problem to remain finite as $t \to \infty$, the coefficient on e^{t} must be zero, thus $y_0 = -\frac{5}{2}$.

Section 2.2: 16

(a) Rewrite the differential equation as $4y^3\,dy = x(x^2+1)\,dx$.
Integrating both sides of the equation results in
$$y^4 = (x^2+1)^2/4 + c.$$
Imposing the initial condition we obtain $c = 0$.
Hence the solution may be expressed as $(x^2+1) - 4y^4 = 0$.
The explicit form of the equation is $y(x) = -\sqrt{(x^2+1)/2}$.
The sign is chosen based on $y(0) = -\frac{1}{\sqrt{2}}$

(b)



(c) Since x^2 will be positive (or zero) for all $x \in \mathbb{R}$, the solution is valid for all $x \in \mathbb{R}$.