D.E. HW #3 selected solutions

Section 3.2: 5

Determine the longest interval in which the given initial value problem is certain to have a unique twice differentiable solution.

$$(t-1)y'' - 3y't + 4y = \sin t, \quad y(-2) = 2, \quad y'(-2) = 1$$

Divide by $(t-1)$ to put in form for Theorem 3.2.1

$$y'' - \frac{3}{t-1} \cdot y' + \frac{4}{t-1} \cdot y = \frac{\sin(t)}{t-1}$$

We find one discontinuity at $t = 1$. Thus, the longest interval that contains the point $t_0 = -2$ is $-\infty < t < 1$.

Section 2.8:

13. Note that $\phi_n(0) = 0$ and $\phi_n(1) = 1, \forall n \geq 1$. Let $a \in (0,1)$. Then $\phi_n(a) = a^n$. Clearly, $\lim_{n \to \infty} a^n = 0$. Hence the assertion is true.

Section 3.2:

23. **Yes.** $y_1'' = -4\cos 2t$; $y_2'' = -4\sin 2t$. $W(\cos 2t, \sin 2t) = 2$.

2.8 #5 a) Picard iterates

$$\phi_0(t) = y_0 = 0$$

$$\phi_1(t) = \int_0^t f(s, y_0(s))\,ds = \int_0^t \left(-\frac{y_0(s)}{2} + s\right) ds = \int_0^t s\,ds = \frac{t^2}{2}$$

$$\phi_2(t) = \int_0^t \left(-\frac{(s^2/2)}{2} + s\right) ds = \int_0^t \left(s - \frac{s^2}{4}\right) ds = \frac{t^2}{2} - \frac{t^3}{12}$$

$$\phi_3(t) = \int_0^t \left(-\frac{1}{2}\left(\frac{s^2}{2} - \frac{s^3}{12}\right) + s\right) ds = \frac{t^2}{2} - \frac{t^3}{12} + \frac{t^4}{96}$$

$$\phi_4(t) = \int_0^t \left(-\frac{1}{2}\left(\frac{s^2}{2} - \frac{s^3}{12} + \frac{s^4}{96}\right) + s\right) ds = \frac{t^2}{2} - \frac{t^3}{12} + \frac{t^4}{96} - \frac{t^5}{960}.$$

Note we can rewrite this as $\frac{t^2}{2} - \frac{1}{2}\frac{t^3}{3!} + \frac{1}{2^2}\frac{t^4}{4!} - \frac{1}{2^3}\frac{t^5}{5!}$

ie, $\phi_4(t) = \frac{t^2}{2} + \sum_{j=3}^{5} \frac{(-t)^j}{2^{j-2} \cdot j!}$

So we conjecture in general that

$$\phi_n(t) = \frac{t^2}{2} + \sum_{j=3}^{n+1} \frac{(-t)^j}{2^{j-2} \cdot j!}$$

We can prove this by induction:

pf

The base step is when $n = 2$, which is $\phi_2 = \frac{t^2}{2} - \frac{t^3}{12}$ and is done above.

Inductive hypothesis: $\phi_k(t) = \frac{t^2}{2} + \sum_{j=3}^{k+1} \frac{(-t)^j}{2^{j-2} \cdot j!}$

inductive claim $\phi_{k+1}(t) = \frac{t^2}{2} + \sum_{j=3}^{k+2} \frac{(-t)^j}{2^{j-2} \cdot j!}$

Inductive step: $\phi_{k+1}(t) = \int_0^t f(s, y_k(s))\, ds$

$$= \int_0^t \left[-\frac{1}{2}\left(\frac{s^2}{2} + \sum_{j=3}^{k+1} \frac{(-s)^j}{2^{j-2}\, j!}\right) + s\right] ds$$

$$= \int_0^t \left[s - \frac{s^2}{4} - \frac{1}{2}\sum_{j=3}^{k+1} \frac{(-s)^j}{2^{j-2}\, j!}\right] ds$$

$$= \int_0^t \left[s - \frac{s^2}{4} + \sum_{j=3}^{k+1} \frac{(-1)^{j+1} s^j}{2^{j-1}\, j!}\right] ds$$

$$= \frac{t^2}{2} - \frac{t^3}{12} + \sum_{j=3}^{k+1}\left(\frac{(-1)^{j+1}}{2^{j-1}\, j!} \int_0^t s^j\, ds\right)$$

$$= \frac{t^2}{2} - \frac{t^3}{12} + \sum_{j=3}^{k+1}\left(\frac{(-1)^{j+1}}{2^{j-1} \cdot j!} \cdot \frac{t^{j+1}}{j+1}\right)$$

$$= \frac{t^2}{2} - \frac{t^3}{12} + \sum_{j=3}^{k+1} \frac{(-t)^{j+1}}{2^{j-1} (j+1)!}$$

Let $i = j+1$ so if $j=3$, $i=4$

$j = i-1$, $j = k+1 \Rightarrow i = k+2$

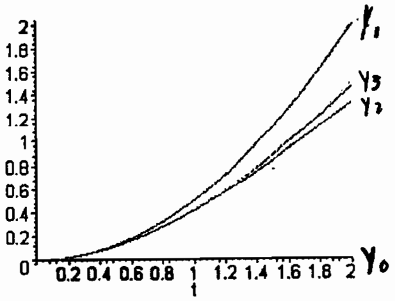
Substitute:

$$= \frac{t^2}{2} - \frac{t^3}{12} + \sum_{i=4}^{k+2} \frac{(-t)^i}{2^{i-2} \cdot i!}$$

Incorporate $\frac{-t^3}{12}$ into the sum to get

$$= \frac{t^2}{2} + \sum_{i=3}^{k+2} \frac{(-t)^i}{2^{i-2} \cdot i!}$$ as required ✓

b)



Yes, they appear to converge.

c) Note that since $e^t = \sum_{j=0}^{\infty} \frac{t^j}{j!}$, we get

$$e^{-t/2} = \sum_{j=0}^{\infty} \frac{(-t/2)^j}{j!} = \sum_{j=0}^{\infty} \frac{(-t)^j}{(2^j)\,j!}$$

and $$4e^{-t/2} = 4\sum_{j=0}^{\infty} \frac{(-t)^j}{2^j\, j!} = \sum_{j=0}^{\infty} \frac{(-t)^j}{2^{j-2}\, j!}$$

$$= 4 - 2t + \frac{t^2}{2} - \cdots$$

So $\frac{t^2}{2} - \sum_{j=3}^{\infty} \frac{(-t)^j}{2^{j-2}\, j!} = 4e^{-t/2} - 4 + 2t$

Thus $y_n \longrightarrow y = 4e^{-t/2} - 4 + 2t$

We can check this is a solution to the IVP:

$y(0) = 4e^{-0/2} - 4 + 2\cdot 0 = 4 - 4 = 0$ ✓

$y' = 4 \cdot \frac{-1}{2} e^{-t/2} + 2 = -2e^{-t/2} + 2$

$f(t,y) = \frac{-y}{2} + t = -\frac{(4e^{-t/2} - 4 + 2t)}{2} + t = -2e^{-t/2} + 2 - t + t$

$= -2e^{-t/2} + 2$

= ✓

we can also see this by solving the equation:

$$y' = -\frac{y}{2} + t$$

$$y' + \frac{y}{2} = t \qquad \mu = e^{t/2}$$

$$(e^{t/2} y)' = e^{t/2} \cdot t$$

so $e^{t/2} y = \int t e^{t/2} \, dt$ let $u = t$ $dv = e^{t/2} dt$

$du = dt$ $v = 2e^{t/2}$

$$e^{t/2} y = 2te^{t/2} - \int 2e^{t/2} \, dt$$

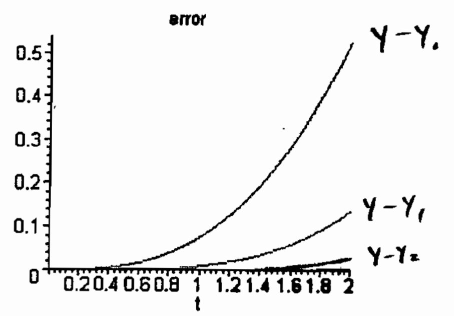
$$e^{t/2} y = 2te^{t/2} - 4e^{t/2} + c$$

so $y = 2t - 4 + ce^{-t/2}$

and if $y(0) = 0$, we get $c = 4$ so

$y = 2t - 4 + 4e^{-t/2}$ is the solution, as we found before.

d)



Y_0 looks reasonable on $t < .2$

Y_1 looks reasonable on $t < .8$

Y_2 looks reasonable on $t < 1.4$

etc.