# DIFFERENTIAL EQUATIONS
# MIDTERM EXAM
# FALL 2000

I. PROBLEMS: DO ALL OF THE FOLLOWING PROBLEMS. YOU MAY USE A CALCULATOR, BUT CLEARLY EXPLAIN WHAT YOU DID AND HOW YOU USED THE CALCULATOR. (6 pts each)

1. SOLVE THE IVP $y' + 2ty = 2te^{-t^2}$, $y(0) = 1$

2. USE EULER'S METHOD WITH $h = .1, .05, .01$, and $.001$ to find an approximation for $y(1)$, where $y$ is the solution to the IVP. (Write down estimate in each case, then guess a value for $y(1)$.) Indicate the accuracy you expect you have based on the data from the calculator.

$$y' = (4 - ty)/(1+y^2) \quad y(0) = -2$$

3. Solve the IVP $9y'' + 6y' + 82y = 0$, $y(0) = -1$, $y'(0) = 2$

II. FORMAL SOLUTIONS: Prove the general form of solutions for one of the following types of equations:
(12 points)

- $1^{st}$ order nonhomogeneous linear
- Exact
- $2^{nd}$ order homogeneous linear with constant coefficients.

clearly indicate which, I will grade only one.

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III. Theorems / definitions: Do one of the following problems (15 points):

1) Define stability, instability, asymptotic stability. Explain these definitions and also semi-stability.
2) State the error formula for Euler's method. Explain Euler's method and the idea behind the error formula.
3) State the existence & uniqueness theorem for 1st order ode's, sketch the proof.
4) State the theorem about the form of solutions for 2nd order homogeneous, linear ode's and prove it.

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IV *'d problems: Do 3 of the following. (15 pts each) clearly indicate which.

3. Consider the differential equation

$$dy/dt = -ay + b,$$

where both $a$ and $b$ are positive numbers.
(a) Solve the differential equation.
(b) Sketch the solution for several different initial conditions.
(c) Describe how the solutions change under each of the following conditions:
 i. $a$ increases.
 ii. $b$ increases.
 iii. Both $a$ and $b$ increase, but the ratio $b/a$ remains the same.

In the following problem, let $\phi_0(t) = 0$ and use the method of successive approximations to solve the given initial value problem. (Picard Iterates)

← you may use the calculator for this

(a) Determine $\phi_n(t)$ for an arbitrary value of $n$.

(c) Express $\lim_{n\to\infty} \phi_n(t) = \phi(t)$ in terms of elementary functions; that is, solve the given initial value problem.

5. $y' = -y/2 + t$, $y(0) = 0$

20. **Convergence of Euler's Method.** It can be shown that, under suitable conditions on $f$, the numerical approximation generated by the Euler method for the initial value problem $y' = f(t, y)$, $y(t_0) = y_0$ converges to the exact solution as the step size $h$ decreases. This is illustrated by the following example. Consider the initial value problem

$$y' = 1 - t + y, \qquad y(t_0) = y_0.$$

(a) Show that the exact solution is $y = \phi(t) = (y_0 - t_0)e^{t-t_0} + t$.
(b) Using the Euler formula, show that

$$y_k = (1+h)y_{k-1} + h - ht_{k-1}, \qquad k = 1, 2, \dots.$$

(c) Noting that $y_1 = (1+h)(y_0 - t_0) + t_1$, show by induction that

$$y_n = (1+h)^n(y_0 - t_0) + t_n \qquad \text{(i)}$$

for each positive integer $n$.

(d) Consider a fixed point $t > t_0$ and for a given $n$ choose $h = (t - t_0)/n$. Then $t_n = t$ for every $n$. Note also that $h \to 0$ as $n \to \infty$. By substituting for $h$ in Eq. (i) and letting $n \to \infty$, show that $y_n \to \phi(t)$ as $n \to \infty$.
*Hint:* $\lim_{n\to\infty}(1 + a/n)^n = e^a$.

23. $y' = \frac{1}{2} - t + 2y$, $\quad y(0) = 1 \quad$ *Hint:* $y_1 = (1 + 2h) + t_1/2$

**19.** Find a solution of the initial-value problem $y' = t\sqrt{1-y^2}$, $y(0) = 1$, other than $y(t) = 1$. Does this violate Theorem 2'? Explain. (Existence & Uniqueness)

← you may use the calculator for this

13. If the functions $y_1$ and $y_2$ are linearly independent solutions of $y'' + p(t)y' + q(t)y = 0$, determine under what conditions the functions $y_3 = a_1y_1 + a_2y_2$ and $y_4 = b_1y_1 + b_2y_2$ also form a linearly independent set of solutions.

14. Consider the sequence $\phi_n(x) = 2nxe^{-nx^2}, 0 \leq x \leq 1$.
(a) Show that $\lim_{n\to\infty} \phi_n(x) = 0$ for $0 \leq x \leq 1$; hence

$$\int_0^1 \lim_{n\to\infty} \phi_n(x)\, dx = 0.$$

(b) Show that $\int_0^1 2nxe^{-nx^2}\, dx = 1 - e^{-n}$; hence

$$\lim_{n\to\infty} \int_0^1 \phi_n(x)\, dx = 1.$$

Thus, in this example,

$$\lim_{n\to\infty} \int_a^b \phi_n(x)\, dx \neq \int_a^b \lim_{n\to\infty} \phi_n(x)\, dx,$$

even though $\lim_{n\to\infty} \phi_n(x)$ exists and is continuous.