MATH 21 FINAL
FALL 2000

HONOR PLEDGE ______________________ NAME ________________________________

THIS IS A 3 HOUR EXAM. PLEASE STAY IN THIS ROOM. THIS EXAM IS WORTH 115 POINTS. IT IS MARKED HERE OUT OF 105. YOUR TOTAL SCORE WILL BE MULTIPLIED BY 1.1. SUGGESTED TIMES ARE GIVEN. OTHER THAN 1 3x5 CARD WITH ESSAY NOTES, NO NOTES OR BOOKS ARE PERMITTED. CALCULATORS ARE ALLOWED, BUT I MUST FIRST CLEAR ALL MEMORY. PLEASE STAPLE YOUR NOTECARD TO YOUR EXAM.

I. <u>ESSAY</u> Explain one of the presentations given
15 pts) by students in class, other than your own group.
45 min) Explain what question was being answered, the mathematical model constructed, and the conclusions it gave about the question. Also explain any special points of the presentation.

II. *'d PROBLEMS (10 pts each, 20 minutes each)

1) Do one of the following two problems:

a) SHOW THAT THE SET OF SOLUTIONS TO A SECOND ORDER LINEAR O.D.E. FORMS A VECTOR SPACE IF AND ONLY IF THE ~~SYSTEM~~ EQUATION IS HOMOGENEOUS.

b) FIND A BASIS FOR THE SET OF POLYNOMIALS OF DEGREE ≤ 4. PROVE IT IS A BASIS.

2) DO ONE of the following two problems:

a) PROVE: CLAIM IF $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ are two solutions of $\vec{x}' = M(t)\vec{x}$ on the interval $\alpha < t < \beta$, then in this interval $W[\vec{x}^{(1)}, \vec{x}^{(2)}]$ is either identically zero or else never vanishes.

(assume $\vec{x}^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix}$, $\vec{x}^{(2)} = \begin{pmatrix} x_1^{(2)} \\ x_2^{(2)} \end{pmatrix}$):

USE OUTLINE:

- Show
$$\frac{dW}{dt} = \begin{vmatrix} \frac{dx_1^{(1)}}{dt} & \frac{dx_1^{(2)}}{dt} \\ x_2^{(1)} & x_2^{(2)} \end{vmatrix} + \begin{vmatrix} x_1^{(1)} & x_1^{(2)} \\ \frac{dx_2^{(1)}}{dt} & \frac{dx_2^{(2)}}{dt} \end{vmatrix}$$

- SHOW $\frac{dW}{dt} = (p_{11} + p_{22})\, W$, where $M(t) = \begin{pmatrix} p_{11} & p \\ p_{21} & \end{pmatrix}$

- Find $W(t)$ by solving this equation. Use the solution to prove the claim.

2) b) Consider the equation $ay''+by'+cy=0$, where a, b, c are constants. In chapter 3, it was shown that the general solution to this equation depended on the roots of the characteristic equation $ar^2+br+c=0$.

- Transform the 1st eqn into a system of first order equations by letting $x_1=y$, $x_2=y'$. Find the system of equations $\vec{x}\,'=A\vec{x}$ satisfied by $\vec{x}=\begin{pmatrix}x_1\\x_2\end{pmatrix}$.

- Find the equation that determines the eigenvalues of the coefficient matrix A. Show this is the same as the characteristic equation above.

3) DO ONE OF THE FOLLOWING TWO problems:

a)

► 18. For small, slowly falling objects the assumption made in the text that the drag force is proportional to the velocity is a good one. For larger, more rapidly falling objects it is more accurate to assume that the drag force is proportional to the square of the velocity.[2]
(a) Write a differential equation for the velocity of a falling object of mass m if the drag force is proportional to the square of the velocity.
(b) Determine the limiting velocity after a long time.
(c) If $m = 10$ kg, find the drag coefficient so that the limiting velocity is 49 m/sec.
(d) Using the data in part (c), draw a direction field and compare it with Figure 1.1.3.

NOTE: YOU NEED NOT SOLVE THE EQUATION TO ANSWER THESE QUESTIONS

b)

► 14. A pond containing 1,000,000 gal of water is initially free of a certain undesirable chemical (see Problem 15 of Section 1.1). Water containing 0.01 g/gal of the chemical flows into the pond at a rate of 300 gal/hr and water also flows out of the pond at the same rate. Assume that the chemical is uniformly distributed throughout the pond.
(a) Let $Q(t)$ be the amount of the chemical in the pond at time t. Write down an initial value problem for $Q(t)$.
(b) Solve the problem in part (a) for $Q(t)$. How much chemical is in the pond after 1 year?
(c) At the end of 1 year the source of the chemical in the pond is removed and thereafter pure water flows into the pond and the mixture flows out at the same rate as before. Write down the initial value problem that describes this new situation.
(d) Solve the initial value problem in part (c). How much chemical remains in the pond after 1 additional year (2 years from the beginning of the problem)?
(e) How long does it take for $Q(t)$ to be reduced to 10 g?
(f) Plot $Q(t)$ versus t for 3 years.

III. DEFINITIONS & THEOREMS (20 min, 15 pts each)

DO BOTH

1) Define eigenvalue, eigenvector and eigenbasis. Explain algebraically and geometrically what they are. How are they relevant to O.D.E.'s?

2) Prove that a linear transformation is determined by its action on a basis. How do we record this?

IV. INSTRUCTOR'S CHOICE (10 pts, 20 minutes)

Draw slope field diagrams and flow lines for autonomous 1st order equations around:

1) a stable point which is not asymptotically stable
2) an asymptotically stable point
3) A semi-stable point
4) An unstable point which is not semi-stable.

Draw flow line diagrams for 2nd order linear homogeneous systems of equations with

5) 2 positive real roots
6) 2 negative real roots
7) 1 positive, 1 negative real root
8) 2 imaginary roots
9) 2 complex roots with positive real part
10) 2 complex roots with negative real part