EXISTENCE AND UNIQUENESS

the extended dance version
with **all** the dirty details. ooo... RACY!

THEOREM Let $f(w,x) : \mathbb{R}^2 \longrightarrow \mathbb{R}$ and $\frac{\partial f}{\partial x} : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be continuous on the rectangle

$$R = \{(w,x) \mid t_0 \le w \le t_0 + a,\ y_0 - b \le x \le y_0 + b\}.$$

Compute $M = \max_R |f(w,x)|$ and set $\alpha = \min\left(a, \frac{b}{M}\right)$.

Then the initial value problem

$$(\ddagger) \quad y'(t) = f(t, y(t)), \quad y(t_0) = y_0$$

has a unique solution on the interval $t_0 \le t \le t_0 + \alpha$.

SOME NOTES 1) There are two functions lurking in the expression $f(t, y(t))$:

$$\begin{array}{ccccc} \mathbb{R} & \xrightarrow{\quad g \quad} & \mathbb{R}^2 & \xrightarrow{\quad f \quad} & \mathbb{R} \\ t & \longrightarrow & (t, y(t)) & & \\ & & (w,x) & \longrightarrow & f(w,x) \end{array}$$

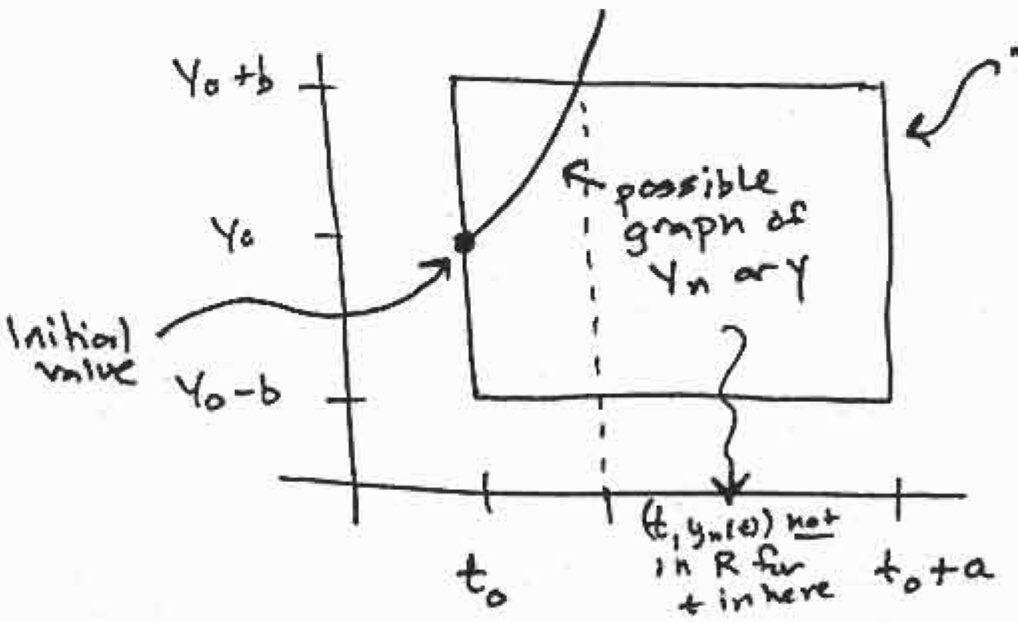
So $f \circ g(t) = f(t, y(t))$.

2) $(\ddagger)$ has a unique solution on $t_0 \le t \le t_0 + \alpha$ means if $(\ddagger)$ has two solutions, say, $y(t)$ and $z(t)$, then $y(t) = z(t)$, $t_0 \le t \le t_0 + \alpha$.

PROOF OUTLINE
1) Explain where α comes from - lemma 1 statemen
2) Uniqueness proof (assuming 4)
3) Construction of Picard Iterates and proof of lemma 1
4) Convergence of Picard Iterates to $y(t)$
5) $y(t)$ is a solution of $(\ddagger)$ on $t_0 \le t \le t_0 + \alpha$.

WHERE α COMES FROM

1) We are going to want to apply the Mean Value Theorem to f at two points in our proof – once in part 2 and once in part 4. Also, we'll want to apply the max min existence theorem to $\frac{\partial f}{\partial x}$. To do this, we need that $\frac{\partial f}{\partial x}$ exists and is continuous in the relevant region. We assume in the hypotheses of this theorem that $\frac{\partial f}{\partial x}$ exists and is continuous on R. So we need to make sure that for the values of t we care about, both $g_n(t) = (t, y_n(t))$ and $(t, y(t)) = g(t)$ are in R.



Here $y_n(t)$ will be the n^{th} Picard iterate. All y_n (and y) must equal y_0 at t_0 (initial value). This is in R, but there is no guarantee tha y_n (and y) don't leave R quickly.

So the point of our first lemma is to find some minimum value $(t_0+\alpha)$, for which we know $(t, y_n(t)) \in R$ and $(t, y(t)) \in R$ for all $t_0 \le t \le t_0+\alpha$.

Lemma 1 Let $y_n(t)$ denote the n^{th} Picard iterate for the initial value problem (‡). Then (for M, α as defined in Thm stmn)

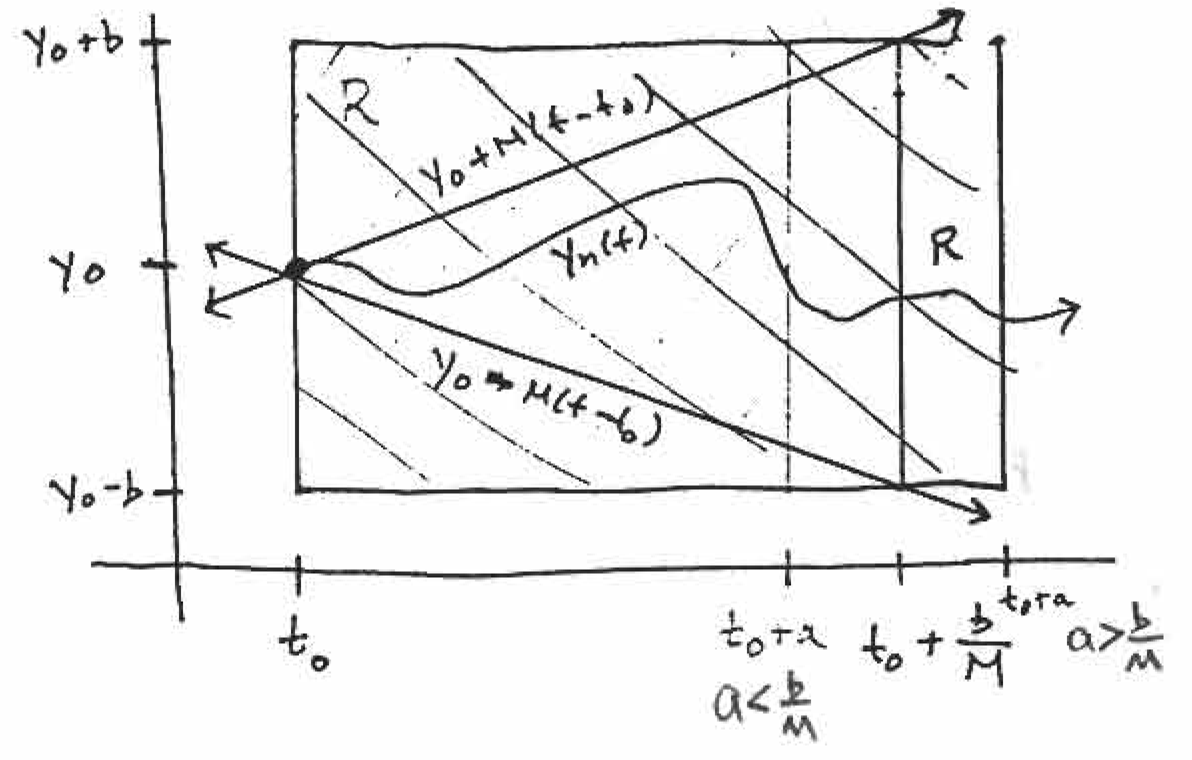
$$|y_n(t) - y_0| \le M(t-t_0) \quad \text{for } t_0 \le t \le t_0+\alpha.$$

The interpretation of lemma 1 is given as follows:

$$\text{if } |y_n(t) - y_0| \le M(t-t_0), \text{ then}$$
$$-M(t-t_0) \le y_n(t) - y_0 \le M(t-t_0), \text{ so}$$
$$y_0 - M(t-t_0) \le y_n(t) \le y_0 + M(t-t_0)$$

That is, the graph of $y_n(t)$ is trapped between the lines:

$$y = y_0 - M(t-t_0) \quad \text{and} \quad y = y_0 + M(t-t_0) \quad \text{for } t_0 \le t \le t_0+\alpha.$$



The line $y_0+M(t-t_0)$ intersects y_0+b at the point $t_0+\frac{b}{M}$

Since $\cancel{y_0}+M(t-t_0) = \cancel{y_0}+b$
$$(t-t_0) = \frac{b}{M}$$
$$t = t_0+\frac{b}{M}$$

Similarly, $y_0-M(t-t_0)$ intersects y_0-b at the point $t_0+\frac{b}{M}$.

So there are two cases. Either 1) $a < \frac{b}{M}$ (R denoted in Red) or 2) $a \geq \frac{b}{M}$ (R denoted in blue)

In case 1), the lines leave R at t_0+a
2) the lines leave R at $t_0+\frac{b}{M}$

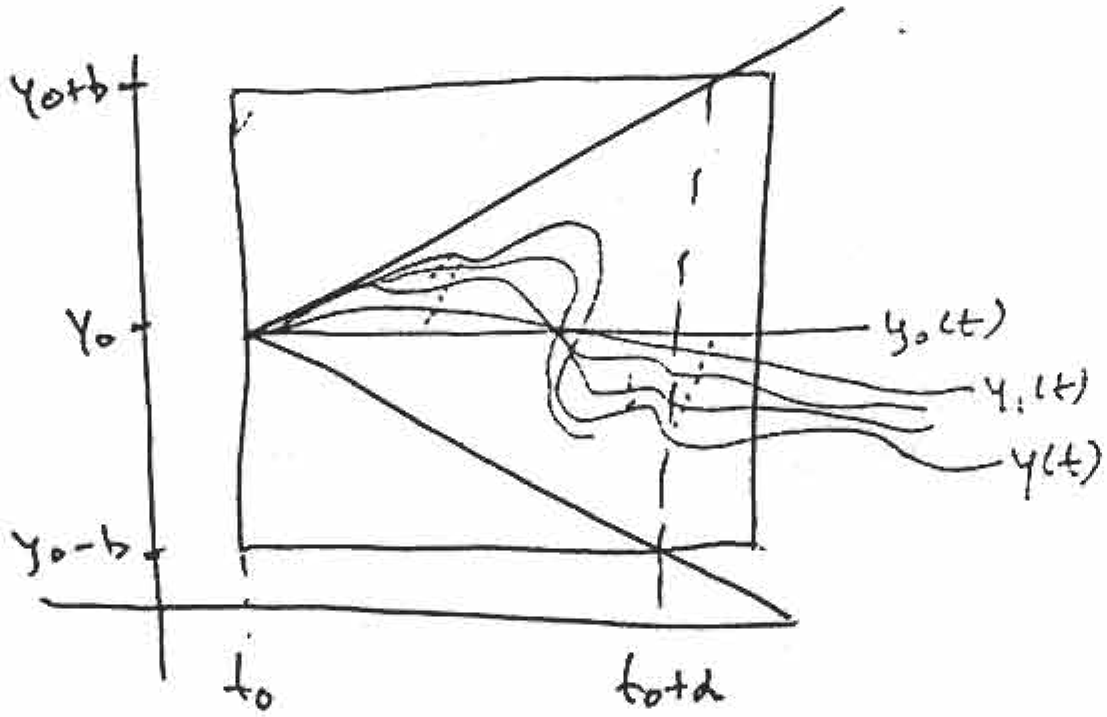
So overall, to ensure $(t, y_n(t))$ is in R, we have to look at the range for t for which it is trapped by the two lines inside R. So we take whichever is less, t_0+a or $t_0+\frac{b}{M}$, and certainly for $t_0<t$ and less than that, $(t, y_n(t))$ will be in R. This is why we set $\alpha = \min(a, \frac{b}{M})$ and only get solution information for
$$t_0 \leq t \leq t_0+\alpha.$$

In part 4 of the proof we'll show that $y_n(t) \xrightarrow{\text{uniformly}} y(t)$, so $y(t)$ is continuous. This also means since the $y_n(t)$ were all trapped in R for $t_0 \leq t \leq t_0+\alpha$, so is $y(t)$. This is important, because we want $(t, y(t))$ inside R for $t_0 \leq t \leq t_0+\alpha$, as well, so we can use the Mean value and Max-Min existence theorems in part 2).

2) Uniqueness

To show that the solution, $y(t)$, to (‡) which we'll get in part 4) is unique, we assume that (‡) has another solution $z(t)$. This means $z(t)$ is continuous, $z(t_0)=y_0$, and $z'(t)=f(t,z(t))$. We want to show $z(t)=y(t)$ on $t_0 \le t \le t_0+$

Now there is one twist in the uniqueness proof which I didn't mention before, and that is again related to the issue of applying the Mean value and Min max existence theorems* to f.
We know from lemma 1 and part 4) that the graph of $y(t)$ is in R for $t_0 \le t \le t_0+\alpha$, since $y(t)$ is the limit of the functions $y_n(t)$, and they all have their graphs in R.



* hereafter to be referred to as MVT and MME

But we don't necessarily know that $z(t)$ is the limit of picard iterates, so we don't a priori know that the graph of $z(t)$ is contained in R for $t_0 \le t \le t_0+\alpha$.

Extension lemma

Here's the trick. Since $z(t)$ is continuous and $z(t_0)=y_0$, we know that $z(t)$ starts inside R!

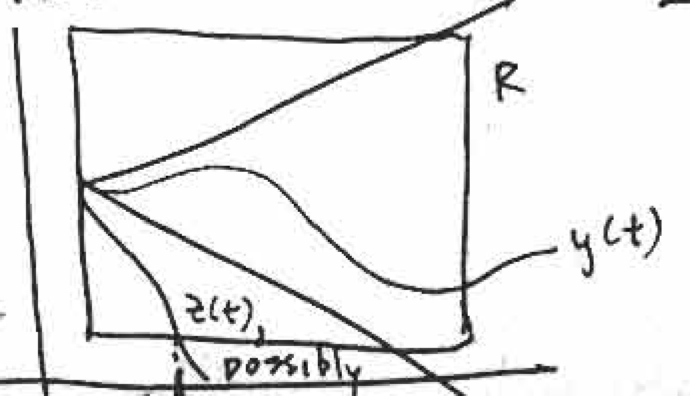


fig 1

So there is some $c_1>0$ such that for $t_0 \le t \le t_0+c_1$, $z(t)$'s graph is in R. (see fig 1)
So there we can use the uniqueness proof from Wednesday.

Since we'll be set for MVT and MME thm. So the uniqueness proof on the interval $t_0 \le t \le t_0 + c_1$ implies that $z(t) = y(t)$ for $t_0 \le t \le t_0 + c_1$.

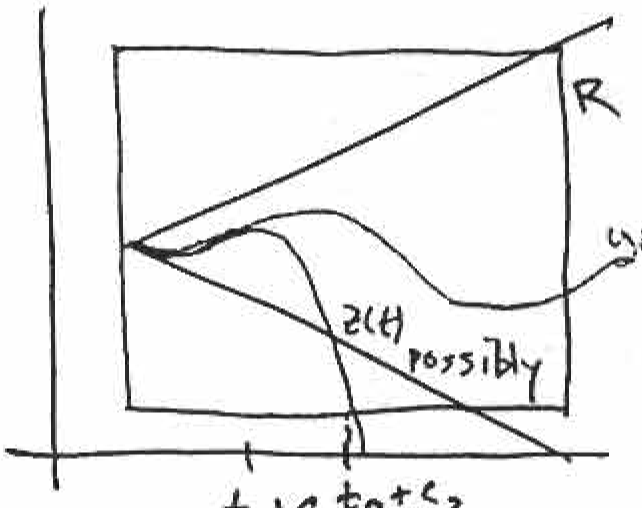


fig 2.

But then, continuity implies that $z(t)$'s graph must be inside R until some later point, $t_0 + c_2$. (see fig 2)

■ end of extension lemma

Then apply the uniqueness lemma to $t_0 \le t \le t_0 + c_2$, to get $z(t) = y(t)$ on $t_0 \le t \le t_0 + c_2$, which will mean $z(t)$ has to be inside R even longer, until $t_0 + c_3$, etc.

So by applying this pair of lemmas (uniqueness and extension, let's call them) over and over again, eventually we get that $z(t) = y(t)$ on the whole interval $t_0 \le t \le t_0 + \alpha$.

Uniqueness lemma Assume that $y(t)$ and $z(t)$ are both solutions to the IVP (‡) on the interval $t_0 \le t \le t_0 + c$, and that $(t, y(t))$ and $(t, z(t))$ are in R for all t in this interval. Then $y(t) = z(t)$ for $t_0 \le t \le t_0 + c$.

pf If $y'(t) = f(t, y(t))$ and $z'(t) = f(t, z(t))$, then integrating these equations in t gives:

$$\int_{t_0}^{t} y'(s)\,ds = \int_{t_0}^{t} f(s, y(s))\,ds \quad \text{and} \quad \int_{t_0}^{t} z'(s)\,ds = \int_{t_0}^{t} f(s, z(s))\,ds$$

$$\Rightarrow \quad y(t) - y(0) = \int_{t_0}^{t} f(s, y(s))\,ds \quad \text{and} \quad z(t) - z(0) = \int_{t_0}^{t} f(s, z(s))\,ds.$$

By initial valve cond $\Rightarrow$ $$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds \quad \text{and} \quad z(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds.$$

We'll get $y(t) = z(t)$ on $t_0 \le t \le t_0 + \alpha$ if

$$|y(t) - z(t)| = 0 \quad \text{for } t_0 \le t \le t_0 + \alpha.$$

So we'll estimate $|y(t) - z(t)|$:

$$|y(t) - z(t)| = \left| \int_{t_0}^{t} \{ f(s, y(s)) - f(s, z(s)) \} \, ds \right|,$$ since the constants y_0 cancel.

$$(*) \qquad \le \int_{t_0}^{t} |f(s, y(s)) - f(s, z(s))| \, ds$$ by the ~~trilangle~~ triangle inequality

Now, since $(s, y(s))$ and $(s, z(s))$ are in R for $t_0 \le s \le t_0 + c$, we can apply the MVT to the function $h(x) = f(s, x)$ on the interval $y(s) \le x \le z(s)$ (or vice versa, $z(s) \le x \le y(s)$). We get that

$$\frac{|h(y(s)) - h(z(s))|}{|y(s) - z(s)|} = h'(\xi_s)$$ for some ξ_s in ~~between~~ between $y(s)$ and $z(s)$.

so

$$(*) \qquad \le \int_{t_0}^{t} |h'(\xi_s)| \, |y(s) - z(s)| \, ds \quad .$$

But also, $h'(\xi_s) = \frac{\partial f}{\partial x}(s, \xi_s)$, and $\frac{\partial f}{\partial x}$ is continuous on R

So by MHE, it attains its max and min. So let $L = \max_R \left| \frac{\partial f}{\partial x}(w, x) \right|$. Then $\left| \frac{\partial f}{\partial x}(s, \xi_s) \right| \le L$, so

$$(*) \qquad \le \int_{t_0}^{t} L \, |y(s) - z(s)| \, ds$$

Overall from page 6, therefore, we get

$$(**) \qquad |y(t) - z(t)| \le \int_{t_0}^{t} L\,|y(s) - z(s)|\,ds .$$

But consider the following:

Let $u(t) = \int_{t_0}^{t} |y(s) - z(s)|\,ds$.

$u(t)$ is positive and increasing, since $|y(s)-z(s)|$ is, and $u(t_0)=0$.

Also $\frac{du}{dt} = |y(s) - z(s)|$ by the fundamental Theorem of Calculus.

(we're seeing <u>all</u> of our favorites

So $0 \le \frac{dy}{dt} = |y(t) - z(t)| \le L \int_{t_0}^{t} |y(s) - z(s)|\,ds = L\,u(t)$.

But if, as we have, $\frac{du}{dt} \le L\,u(t)$, then

$$\left(e^{-L(t-t_0)}u(t)\right)' = -L e^{-L(t-t_0)}u(t) + e^{-L(t-t_0)}u'(t)$$

$$\le -L e^{-L(t-t_0)}u(t) + e^{-L(t-t_0)}L\,u(t)$$

$$= 0 .$$

So $e^{-L(t-t_0)}u(t)$ is decreasing. But

1) $u(t)$ is positive (≥ 0), and so is $e^{-L(t-t_0)}$
2) $u(t_0) = 0$

So $0 \le e^{-L(t-t_0)}u(t) \le e^{-L(t_0-t_0)}u(t_0) = 0$. This can only happen if $u(t) \equiv 0$ (is <u>identically</u> equal to).

putting this together with estimate (**) gives

$$0 \le |y(t)-z(t)| \le \int_{t_0}^{t} L\,|y(s)-z(s)|\,ds = L\,U(t) = 0$$

so $|y(t)-z(t)|$ on $t_0 \le t \le t_0+c$, and we are done with the existence Lemma (<u>finally</u>). ■

3) CONSTRUCTION OF PICARD ITERATES

Recall from 2) that our initial value problem, (‡), can be rewritten in the integral form

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\,ds .$$

So we define an operator $L(\varphi(t)) = y_0 + \int_{t_0}^{t} f(s, \varphi(s))\,ds$.

Then $y(t)$ is a solution to (‡) if and only if it is fixed by the operator L, that is, $L(y(t)) = y(t)$.

The <u>idea</u>, then, is to find such a $y(t)$ by starting with a "first guess", $y_0(t) = y_0$ for all t, and taking

$$\text{"}L^{\infty}(y_0(t))\text{"} = \text{the limit of } \underbrace{L(L(\cdots\cdots L}_{n \text{ times}}(y_0(t))\cdots)) \text{ as } n \to \infty$$

Then $L\left(L^{\infty}(y_0(t))\right) = L^{\infty+1}(y_0(t)) = L^{\infty}(y_0(t))$, so $L^{\infty}(y_0(t)) = y(t)$ is the fixed point we want. Of course, we need to justify both that $L^{\infty}(y_0(t))$ makes sense and that $L(L^{\infty}(y_0(t))) = L^{\infty}(y_0(t))$ actually. These are steps 4 and 5 of the proof, respectively.

The picard iterates for (‡) are $y_0(t)$, $\underbrace{L(y_0(t))}_{= y_1(t)}$, $\underbrace{L(L(y_0(t)))}_{L(y_1(t)) = y_2(t)}$, etc,

with $y_n(t) = L(y_{n-1}(t))$

$$= y_0 + \int_{t_0}^{t} f(s, y_{n-1}(s))\,ds$$

Notice two things about $y_n(t)$ for all n:

1) $y_n(t_0) = y_0$

2) $y_n(t)$ is the integral of a continuous function, therefore also continuous ($y_0(t)$ is a constant, so continuous, which starts the implied induction in this statement).

Now that we know what $y_n(t)$ are, we need to go back and prove lemma 1.

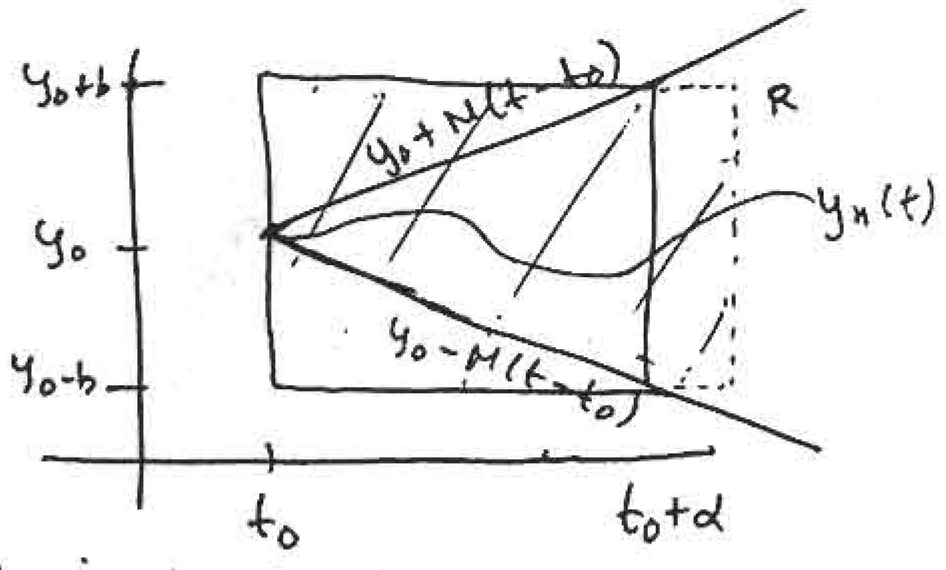
<u>Lemma 1</u> Let $y_n(t)$ denote the n^{th} Picard iterate for the initial value problem ($\dagger$). Then for

$$M = \max_R |f(w,x)| \quad \text{and} \quad \alpha = \min\left(a, \frac{b}{M}\right),$$

we get $|y_n(t) - y_0| \leq M(t-t_0)$ for $t_0 \leq t \leq t_0 + \alpha$.

<u>pf</u>

Recall the picture:



We'll prove this by induction on n:

$n=0$:

$$|y_0(t) - y_0| = 0 \leq M(t-t_0) \checkmark$$

assume for n, show for $n+1$:

$$|y_{n+1}(t) - y_0| = \left|\int_{t_0}^{t} f(s, y_n(s))\, ds\right| \leq \int_{t_0}^{t} |f(s, y_n(s))|\, ds \quad (+)$$

But $(s, y_n(s)) \in R$ by the assumption that ...
$|y_n(t) - y_0| \leq M(t-t_0)$ for $t_0 \leq t \leq t_0+\alpha$. (see picture)

So $|f(s, y_n(s))| \leq M = \max_R |f(w,x)|$.

So $(\dagger) \leq \int_{t_0}^{t} M\, ds = M(t-t_0)$, and we are done. ∎

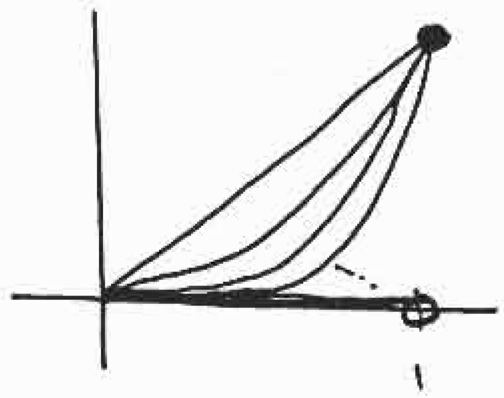
4) CONVERGENCE OF PICARD ITERATES

We wanted to show $y_n(t) \longrightarrow y(t)$ in some suitable sense. What sort of a sense is that? Well, remember that for $y(t)$ to be a solution of $(\ddagger)$, it has to be continuous on $t_0 \le t \le t_0+\alpha$. We know that $y_n(t)$ are all continuous, and we'd like to be able to say that implies $y(t)$ is.

The most obvious type of convergence of functions is <u>pointwise convergence</u>. This means $\lim_{n\to\infty} y_n(t) = y(t)$

for all points $t_0 \le t \le t_0+\alpha$. But a pointwise limit of continuous functions is not necessarily continuous. Consider:

$\psi_n(t) = t^n$ on $[0,1]$. pointwise, $\psi_n(t) \longrightarrow \psi(t) = \begin{cases} 0 & t<1 \\ 1 & t=1 \end{cases}$



So we want a stronger type of convergence, called <u>uniform convergence</u>. I won't define that here, but you should look it up. The important property is that if a sequence of continuous functions converges uniformly to a limit function, then that limit function is also continuous.

There is an important theorem that tells when a <u>series</u> of functions converges uniformly. It is used also (besides ODE's) to show Taylor series converge to continuous functions. Here it is: (no proof here)

<u>Weierstrass M-test</u>: Let $\{w_n(t)\}$ be a sequence of functions defined on an interval, I. Let $\{B_n\}$ be a sequence of numbers such that
$$|w_j(t)| \le B_j \quad \text{for all } t \in I.$$
Suppose that $\sum_{j=1}^{\infty} M_j$ converges. Then for all $t \in I$, $\sum_{j=1}^{\infty} w_j(t)$ converges absolutely, and the partial sums $\sum_{j=1}^{n} w_j(t)$ converge uniformly to $w(t) = \sum_{j=1}^{\infty} w_j(t)$.

We want to use this theorem to show that the functions $y_n(t)$ converge uniformly to $y(t)$. So we need to rewrite $y_n(t)$ as the n^{th} partial sum of some series of functions. By the old add 'n' subtract trick,

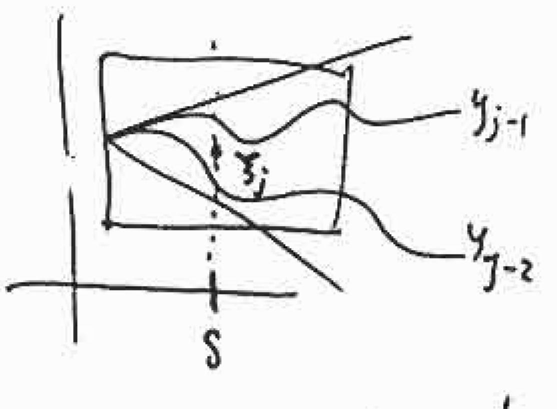
$$y_n(t) = y_0 + (y_1(t) - y_0) + (y_2(t) - y_1(t)) + \cdots + (y_n(t) - y_{n-1}(t))$$

So $w_j = y_j - y_{j-1}$, and $y_n(t) = y_0 + \sum_{j=0}^{n} w_j(t)$.

So we need to find our constants B_j:

$$|w_j(t)| \le |y_j(t) - y_{j-1}(t)| = \left| \int_{t_0}^{t} \{ f(s, y_{j-1}(s)) - f(s, y_{j-2}(s)) \} ds \right|$$
$$\le \int_{t_0}^{t} | f(s, y_{j-1}(s)) - f(s, y_{j-2}(s)) | \, ds$$

Applying the MVT as in 2),
$$= \int_{t_0}^{t} \left| \frac{\partial f}{\partial x}(s, \xi_j(s)) \right| \, |y_{j-1}(s) - y_{j-2}(s)| \, ds$$
for some ξ_j between $y_{j-1}(s)$ and $y_{j-2}(s)$.



$$\le \int_{t_0}^{t} L |y_{j-1}(s) - y_{j-2}(s)| \, ds,$$

where again, $L = \max_R \left| \frac{\partial f}{\partial x}(w, x) \right|$.

so $|y_j(t) - y_{j-1}(t)| \le L \int^{t} |y_{j-1}(s) - y_{j-2}(s)| \, ds$.

So, once again by induction, we get

base: $j=\phi$: $|y_j(t) - y_{j-1}(t)| = |y_1(t) - y_0(t)| \le M(t-t_0)$ by Lemma 1.

inductive step: Assume $|y_j(t) - y_{j-1}(t)| \le M L^{j-1} (t-t_0)^j / j!$.

Show $|y_{j+1}(t) - y_j(t)| \le M L^j (t-t_0)^{j+1} / (j+1)!$:

$$|y_{j+1}(t) - y_j(t)| \le \int_{t_0}^{t} L\, |y_{j-1}(s) - y_{j-2}(s)|\, ds \quad \text{by the last page,}$$

$$\le \int_{t_0}^{t} L \left(M L^{j-1} (s-t_0)^j / j! \right) ds$$

$$= M L^j \frac{(t-t_0)^{j+1}}{(j+1)!} \quad \checkmark$$

So overall, $|y_j(t) - y_{j-1}(t)| \le M L^j \frac{(t-t_0)^{j+1}}{(j+1)!} \le M L^j \frac{\alpha^{j+1}}{(j+1)!}$ since $t - t_0 \le \alpha$. ($t \le t_0 + \alpha$)

So our $B_j = \frac{M}{L} \frac{(L\alpha)^{j+1}}{j+1}$.

But by the ratio test, we get that

$\sum_{j=1}^{\infty} \frac{M}{L} \frac{(L\alpha)^{j+1}}{j+1}$ converges, so by the Weierstrass M-test,

$y_n = \left(\sum_{j=1}^{n} w_j\right) + y_0$ converges uniformly to $y = \left(\sum_{j=1}^{\infty} w_j\right) + y_0$.

5) $y(t)$ is a solution to $(\ddagger)$ on $t_0 \le t \le t_0 + \alpha$

We defined the Picard iterates by

$$(\ddagger) \qquad y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, y_n(t))\,ds \quad t_0 \le t \le t_0 + \alpha.$$

We want to show that $y(t) = y_0 + \int_{t_0}^{t} f(s, y(t))\,ds$, since this is equivalent to $(\ddagger)$.

In 4), we saw that $\lim_{n \to \infty} y_n(t) = y(t)$, so, taking limits of both sides of $(\ddagger)$, we get

$$y(t) = \lim_{n \to \infty} \left(y_0 + \int_{t_0}^{t} f(s, y_n(s))\,ds \right)$$

so

$$y(t) = y_0 + \lim_{n \to \infty} \int_{t_0}^{t} f(s, y_n(s))\,ds.$$

We'd like to pass the limit through the integral, since then (because $f(s, x)$ is continuous on R, so preserves limits), we'd get

$$\int f(s, y(s))\,ds.$$

We can't just do that, though. We really need to show

$$\lim_{n \to \infty} \left| \int_{t_0}^{t} f(s, y(s))\,ds - \int_{t_0}^{t} f(s, y_n(s))\,ds \right| = 0.$$

We proceed along familiar lines...

$$\left| \int_{t_0}^{t} f(s, y(s))\,ds - \int_{t_0}^{t} f(s, y_n(s))\,ds \right| = \left| \int_{t_0}^{t} \{ f(s, y(s)) - f(s, y_n(s)) \}\,ds \right|$$

$$\le \int_{t_0}^{t} \left| f(s, y(s)) - f(s, y_n(s)) \right| ds \quad \text{by triangle inequality}$$

$$= \int_{t_0}^{t} \left| \frac{\partial f}{\partial x}(s, \xi_n) \right| \left| y(s) - y_n(s) \right| ds \quad \text{by MVT}$$

$$\leq \int_{t_0}^{t} L\,|y(s)-y_n(s)|\,ds\,, \qquad L=\max_R \left|\frac{\partial f}{\partial x}\right| \text{ by MME}$$

Since $y(s) = y_0 + \sum_{j=1}^{\infty} y_j(s)-y_{j-1}(s)$, and $y_n(s) = y_0 + \sum_{j=1}^{n} y_j(s)-y_{j-1}(s)$,

$$|y(s)-y_n(s)| = \left|\sum_{j=n+1}^{\infty} y_j(s)-y_{j-1}(s)\right|$$

$$\leq \sum_{j=n+1}^{\infty} |y_j(s)-y_{j-1}(s)| \quad \text{by triangle inequality}$$

$$\leq \sum_{j=n+1}^{\infty} \frac{L^{j-1}M(s-t_0)^j}{j!} \quad \text{by our estimate from 4)}$$

so $\displaystyle\int_{t_0}^{t} L\,|y(s)-y_n(s)|\,ds \leq \int_{t_0}^{t} L\cdot\sum_{j=n+1}^{\infty}\frac{L^{j-1}M(s-t_0)^j}{j!}\,ds$

Since the sum converges uniformly, we can pass the integral through the sum (another useful property of uniformly convergent series) to get

$$\leq \sum_{j=n+1}^{\infty}\frac{L^jM}{j!}\int_{t_0}^{t}(s-t_0)^j\,ds = \sum_{j=n+1}^{\infty}\frac{L^jM\,(s-t_0)^{j+1}}{j+1\,!}$$

So altogether so far we have

$$0 \leq \lim_{n\to\infty}\left|\int_{t_0}^{t} f(s,y(s))\,ds - \int_{t_0}^{t} f(s,y_n(s))\,ds\right| \leq \lim_{n\to\infty}\sum_{j=n+1}^{\infty}\frac{L^jM\,(s-t_0)^{j+1}}{(j+1)!}$$

Since the entire sum $\displaystyle\sum_{j=1}^{\infty}\frac{L^jM\,(s-t_0)^{j+1}}{(j+1)!}$ converges by the ratio test,

the "tail" of the sum, $\displaystyle\sum_{j=n+1}^{\infty}\frac{L^jM\,(s-t_0)^{j+1}}{(j+1)!} \longrightarrow 0$ as $n\to\infty$.

So, by the squeeze theorem, $\displaystyle\lim_{n\to\infty}\left|\int^{t} f(s,y(s))\,ds - \int^{t} f(s,y_n(s))\,ds\right| = 0$

and we have completed the proof that

$$y(t) = \lim_{n\to\infty} y_n(t) = \lim_{n\to\infty}\left(y_0 + \int_{t_0}^{t} f(s, y_n(s))\,ds\right)$$

$$= \quad y_0 + \int_{t_0}^{t} f(s, y(s))\,ds \,, \quad \text{that is,}$$

$y(t)$ is a solution of (‡) on $t_0 \le t \le t_0 + \alpha$.

∎