Fall 2003 Differential Equations
Midterm EXAM SOLUTIONS

1) Thm (Existence & Uniqueness) Let $F(t,y)$ and $F_y(t,y)$ be continuous on the rectangle

$$R=[t_0, t_0+a]\times[y_0-b, y_0+b].$$

Let

$$M=\max_R |F(t,y)| \quad \text{and} \quad \alpha=\min\left(a, \frac{b}{M}\right).$$

Then the IVP $\begin{cases} y'=F(t,y) \\ y(t_0)=y_0 \end{cases}$ has a unique solution on the interval $t_0\le t\le t_0+\alpha$.

2) a) Thm (Exact Equations) Let the functions M, N, M_y and N_x, where the subscripts denote partial derivatives, be continuous in the rectangular region

$$R=(\alpha,\beta)\times(\gamma,\delta).$$

Then $M(x,y)+N(x,y)y'=0$ is an exact differential equation in R if and only if

$$M_x(x,y)=N_x(x,y) \qquad (\ddagger)$$

at each point of R. That is, there exists a function ψ satisfying

$$\psi_x(x,y)=M(x,y) \quad \text{and} \qquad (**)$$
$$\psi_y(M,y)=N(x,y)$$

if and only if M and N satisfy $(\ddagger)$.

b) proof 1) If such a ψ exists, then $(\ddagger)$ must hold:

If $\psi_x(x,y)=M(x,y)$ then $\frac{d}{dy}\psi_x(x,y)=M_y(x,y)$.

If $\psi_y(x,y)=N(x,y)$ then $\frac{d}{dx}\psi_y(x,y)=N_x(x,y)$.

But $\frac{\partial}{\partial y}\psi_x(x,y)=\frac{\partial^2\psi}{\partial y\partial x}=\frac{\partial^2\psi}{\partial x\partial y}=\frac{\partial}{\partial x}\psi_y(x,y)$, since these are continuous, so $M_y(x,y)=N_x(x,y)$.

2) If (‡) holds then ψ exists:

If such a ψ is to exist, then it must satisfy

$$\psi_x(x,y) = M(x,y)$$

so

$$\psi(x,y) = \int M(x,y)\,dx + h(y)$$

for some function h constant in x, thus only depending on y.

Since $\psi_y(x,y) = N(x,y)$ we then need that

$$\frac{\partial}{\partial y}\psi(x,y) = \frac{\partial}{\partial y}\left[\int M(x,y)dx + h(y)\right] = N(x,y)$$

Since M_y is continuous, we can bring the derivative inside (somewhat more subtle than this really), to get

$$\int M_y(x,y)\,dx + h'(y) = N(x,y).$$

Thus $h'(y) = N(x,y) - \int M_y(x,y)\,dx$.
If such an h is to exist, we need to know that the right side, which depends a priori both on x and on y, in fact only depends on y. To show this, we differentiate with respect to x to get:

$$\frac{\partial}{\partial x}\left[N(x,y) - \int M_y(x,y)\,dx\right] = N_x(x,y) - M_y(x,y) = 0.$$

by the fundamental theorem of calculus and by (‡).

Thus we can find such an h:

$$h(y) = \int [N(x,y) - \int M_y(x,y)\,dx]\,dy$$

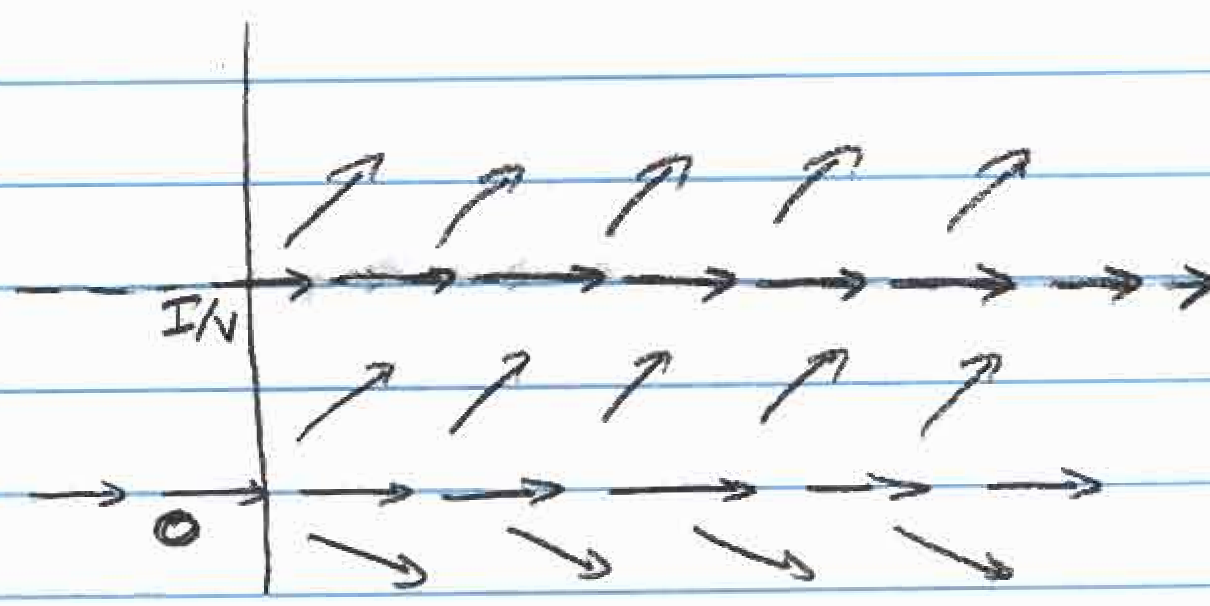
And we get the required function

$$\Psi(x,y) = \int M(x,y)\,dx + \int [N(x,y) - \int M_y(x,y)\,dx]\,dy,$$

and we are done. ∎

3) a) Steady states: $0 = V' = \lambda V (I - \alpha V)^2$
if $V = 0$ or if $I - \alpha V = 0$, that is,
if $V = 0$ or $V = \frac{I}{V}$.

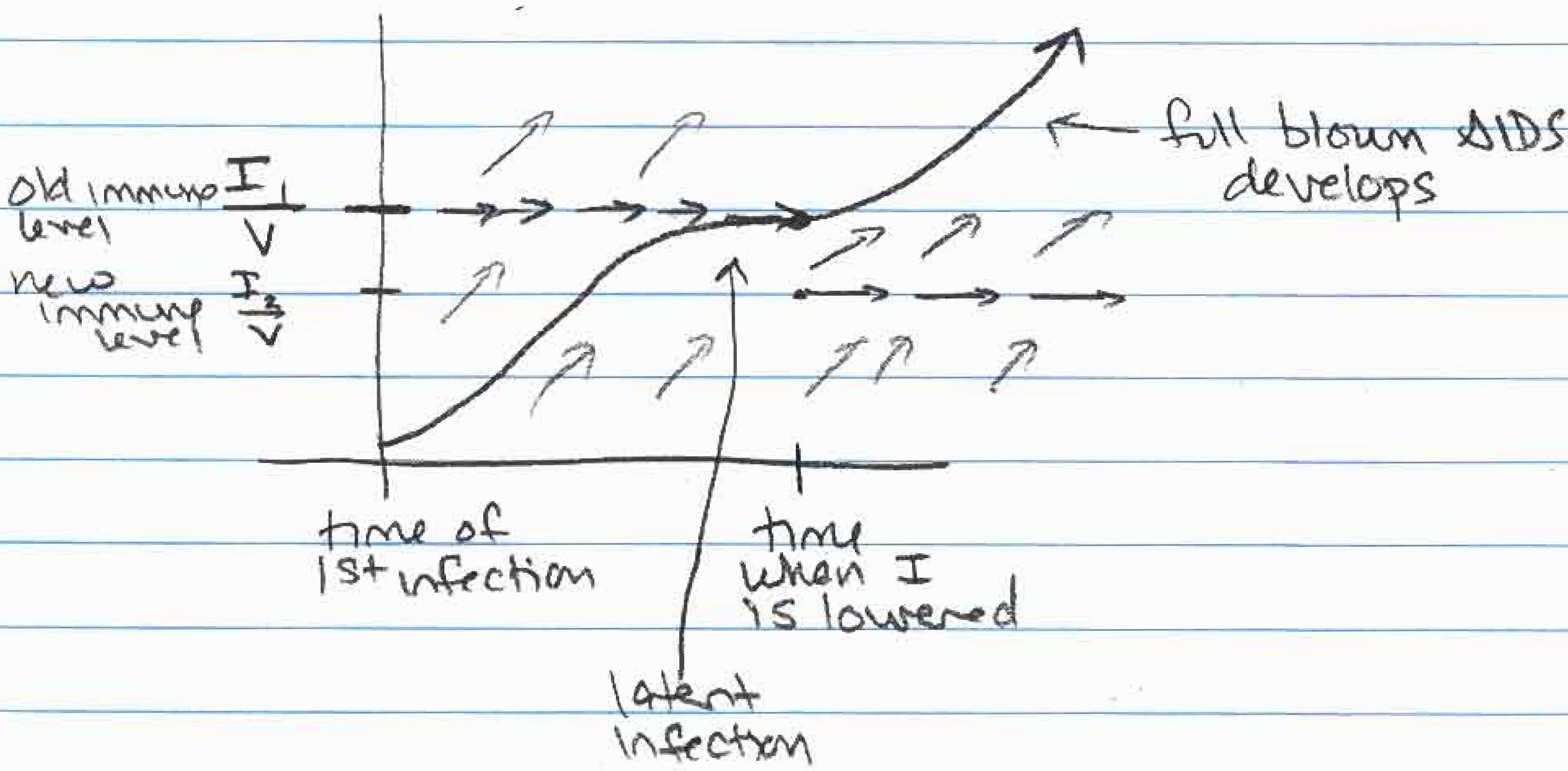
Slope field diagram:



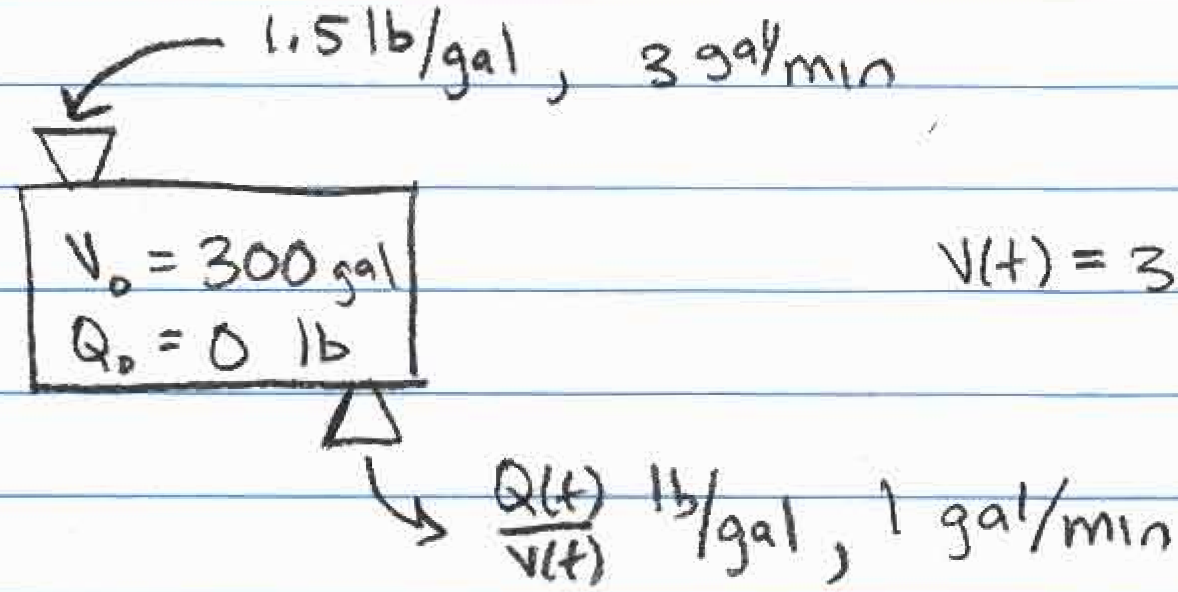
Stability: 0 is an unstable solution
$\frac{I}{V}$ is a semi-stable solution

b) The semi-stable state $\frac{I}{V}$ represents the latently infected state. When an individual is infected, the virus reaches a certain level in the body and stays there until some perturbation moves it to the unstable side.

c) If I decreases, $\frac{I}{V}$ decreases. If I increases, $\frac{I}{V}$ increases. If a latently infected individual has a reduction in immune response due to some other factor, then the steady state may fall below the current level of infection, which is then in the unstable and exponentially increasing range where full-blown AIDS develops:



4) a)



$V(t) = 300 + 2t$

b) $Q(t)$ should satisfy the IVP:

$$\begin{cases} Q'(t) = 1.5 \times 3 - \dfrac{Q(t)}{300+2t} \cdot 1 \\ Q(0) = 0 \end{cases}$$

c) Capacity is 600 gal, which happens at $t=150$.

$$Q' + \frac{1}{300+2t} Q = 4.5$$

$$\mu Q' + \mu \frac{1}{300+2t} Q = \mu\, 4.5$$

$$[\mu Q]' = \mu Q' + \mu' Q$$

$$= \mu Q' + \frac{\mu}{300+2t} Q$$

So $\mu' = \frac{\mu}{300+2t}$. Thus $\frac{\mu'}{\mu} = \frac{1}{300+2t}$

so $\int \frac{\mu'}{\mu} dt = \int \frac{1}{300+2t} dt$

$$\int \frac{d\mu}{\mu} = \int \frac{1}{u} \cdot \frac{1}{2} du \qquad u = 300+2t \qquad du = 2dt$$

$$\ln|\mu| = \frac{1}{2} \ln|u|$$

$$\ln|\mu| = \ln\sqrt{u}$$

$\mu = C\sqrt{300+2t}$ will work for any $C \neq 0$

choose $C=1$.

So

$$[\sqrt{300+2t}\, Q]' = 4.5\sqrt{300+2t}$$

so

$$\sqrt{300+2t}\, Q = \int 4.5\sqrt{300+2t}\, dt$$

$$\sqrt{300+2t}\, Q = \frac{4.5}{2} \int \sqrt{u}\, du \qquad u = 300+2t, \quad du = 2dt$$

$$\sqrt{300+2t}\, Q = \frac{4.5}{2} u^{3/2} \cdot \frac{2}{3} + C$$

$$(300+2t)^{1/2} Q = 1.5(300+2t)^{3/2} + C$$

So $Q = 1.5(300+2t) + \frac{c}{\sqrt{300+2t}}$

$Q = 450 + 3t + \frac{c}{\sqrt{300+2t}}$

To find c, use IVP

$0 = Q(0) = 450 + 0 + \frac{c}{\sqrt{300}}$

$c = -450\sqrt{300}$

So $Q = 450 + 3t - \frac{450\sqrt{300}}{\sqrt{300+2t}}$

So $Q(150) = 450 + 3\cdot 150 - \frac{450\sqrt{300}}{\sqrt{300+300}}$

$= 900 - \frac{450\cancel{\sqrt{300}}}{\sqrt{2}\cancel{\sqrt{300}}}$

$= 900 - \frac{450}{\sqrt{2}} \approx 581.802$ lbs when the tank is at capacity

5) Do this essay on your own - if you have questions about how your essay could be improved, bring it by and we can discuss it.

6) <u>Def</u> If $\{\vec{v}_i\} \subset V$ spans the vector space V and is linearly independent, we call $\{\vec{v}_i\} \subset V$ a <u>basis</u> for V.

<u>Def/Thm</u> Any two bases for a vector space V have the same number of elements, called the <u>dimension</u> of V

7) Thm If $p(t)$, $q(t)$ and $g(t)$ are continuous on the interval I, then the IVP

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t) \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}$$

has a unique solution on I, and this solution exists on the whole interval I.

8) a) Def A homogeneous second order linear o.d.e is an equation of the form

$$y'' + p(t)y' + q(t)y = 0.$$

b) Def A homogeneous linear differential equation or system is one whose solution set forms a vector space.

c) To show $V = \{y \mid y'' + p(t)y' + q(t)y = 0\}$ is a vector space, we first note that $V \subset \mathcal{F}(I)$ where I is the interval of existence guarenteed by the existence and uniqueness theorem. Since $\mathcal{F}(I)$ is a vector space, we only need to show V is a subspace of it. By the subspace theorem, this means we only have to verify that V is closed under vector (function) addition and scalar multiplication.

Closure under +.

Suppose y_1 and y_2 are both in V.

Then we know that

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \qquad (*)$$

and $$y_2'' + p(t)y_2' + q(t)y_2 = 0$$

Adding these together, we get the equation:

$$(y_1'' + y_2'') + (p(t)y_1' + p(t)y_2') + (q(t)y_1 + q(t)y_2) = 0$$

By distributivity and the linearity of derivatives, this is equivalent to:

$$(y_1+y_2)'' + p(t)(y_1+y_2)' + q(t)(y_1+y_2) = 0$$

which says exactly that $y_1+y_2 \in V$, too. Hence V is closed under vector addition.

Closure under $\cdot$: If $y_1 \in V$ and $a \in \mathbb{R}$, then multiplying $(*)$ by a, we get:

$$ay_1'' + ap(t)y_1' + aq(t)y_1 = 0$$

By commutativity and by linearity of the derivative, this is equivalent to

$$(ay_1)'' + p(t)(ay_1)' + q(t)(ay_1) = 0$$

Which says exactly that $ay_1 \in V$, too. Hence V is closed under scalar multiplication.

Thus $V \subset \mathcal{F}(I)$ is a subspace, thus a vector space in its own right. ∎

9) a) $y'' + 8y' - 9y = 0$ $\quad (*)$

$\lambda^2 + 8\lambda - 9 = 0$

$(\lambda+9)(\lambda-1) = 0$ $\quad$ so $\lambda = 1, -9$

Thus $y_1 = e^t$ and $y_2 = e^{-9t}$ are two solutions to $(*)$.

b) If $y = c_1 y_1 + c_2 y_2$, then

$y = c_1 e^t + c_2 e^{-9t}$ $\quad y(0) = c_1 + c_2 = 12$

So $y' = c_1 e^t - 9c_2 e^{-9t}$ $\quad -y'(0) = c_1 - 9c_2 = -8$

So we need $\begin{pmatrix} 1 & 1 \\ 1 & -9 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 12 \\ -8 \end{pmatrix}$

Invert:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \frac{1}{1\cdot -9 - 1\cdot 1}\begin{pmatrix} -9 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 12 \\ 8 \end{pmatrix} = \frac{-1}{10}\begin{pmatrix} -9 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 12 \\ -8 \end{pmatrix}$$

$$= \frac{1}{10}\begin{pmatrix} 9 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 12 \\ -8 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 9\cdot 12 - 8 \\ 12 + 8 \end{pmatrix} = \begin{pmatrix} 10 \\ 2 \end{pmatrix}$$

So $y = 10e^t + 2e^{-9t}$ solves this IVP.

10) a) $y_1 = \cos(\ln t)$

$y_1' = -\sin(\ln t)\cdot \frac{1}{t} = -\frac{1}{t}\sin(\ln t)$

$y_1'' = \frac{1}{t^2}\sin(\ln t) - \frac{1}{t}\cos(\ln t)\cdot\frac{1}{t}$

$= \frac{1}{t^2}\sin(\ln t) - \frac{1}{t^2}\cos(\ln t)$

so

$t^2 y_1'' + t y_1' + y_1 = t^2\left[\frac{1}{t^2}\sin(\ln t) - \frac{1}{t^2}\cos(\ln t)\right] + t\left[-\frac{1}{t}\sin(\ln t)\right] + \cos(\ln t)$

$= \sin(\ln t) - \cos(\ln t) - \sin(\ln t) + \cos(\ln t) = 0$

So y_1 is a solution.

$y_2 = \sin(\ln t)$

$y_2' = \frac{1}{t}\cos(\ln t)$

$y_2'' = -\frac{1}{t^2}\cos(\ln t) - \frac{1}{t}\sin(\ln t)\cdot\frac{1}{t}$

$= -\frac{1}{t^2}\cos(\ln t) - \frac{1}{t^2}\sin(\ln t)$

So $t^2 y_2'' + t y_2' + y_2 = t^2\left[-\frac{1}{t^2}\cos(\ln t) - \frac{1}{t^2}\sin(\ln t)\right]$

$+ t\left[\frac{1}{t}\cos(\ln t)\right] + \sin(\ln t)$

$= -\cos(\ln t) - \sin(\ln t) + \cos(\ln t) + \sin(\ln t)$

$= 0$

Thus y_2 is also a solution.

b) To check they form a basis for the solution space (or fundamental solution set) we calculate their Wronskian;

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos(\ln t) & \sin(\ln t) \\ -\frac{1}{t}\sin(\ln t) & \frac{1}{t}\cos(\ln t) \end{vmatrix}$$

$= \frac{1}{t}\cos^2(\ln t) + \frac{1}{t}\sin^2(\ln t) = \frac{1}{t} \neq 0$ on the

interval $t > 0$ for which $p(t) = \frac{1}{t}$ and $q(t) = \frac{1}{t^2}$ are continuous.

($t^2 y'' + t y' + y = 0$ rearranges to $y'' + \frac{1}{t}y' + \frac{1}{t^2}y = 0$)

Thus y_1 and y_2 form a basis.