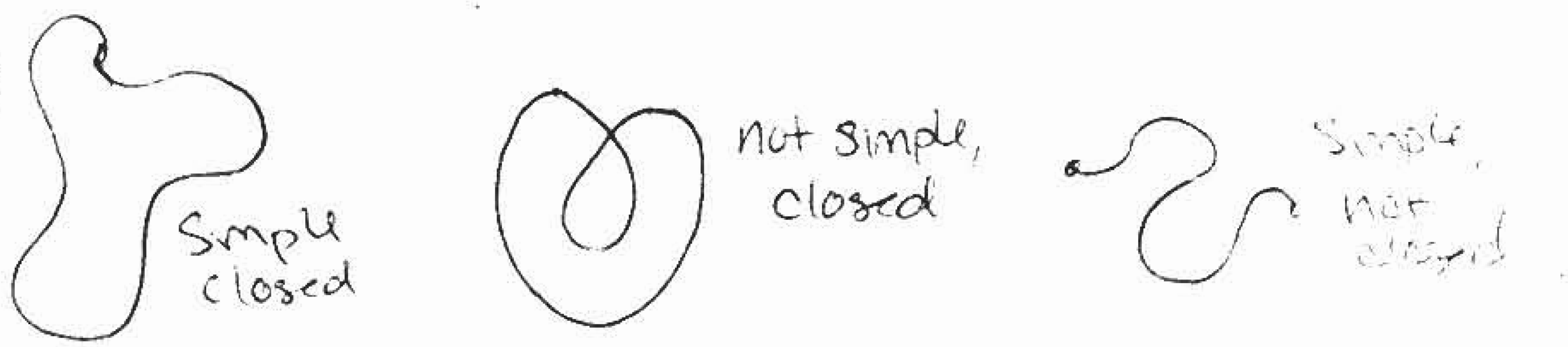
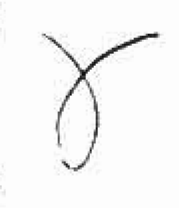
PRACTICE EXAM SOLUTIONS

I, II – If you have sample answers to these essay questions or any other, I'd be happy to look over them with you today (Sunday) after the review if you are taking the exam Monday, or tomorrow 2 – 4:30 if you are taking it Tuesday at 1:30.

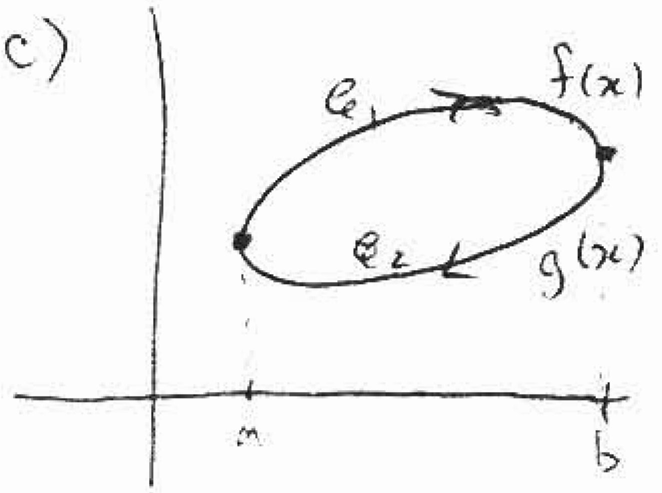
III. a) A simple closed curve is a curve $\mathcal{C}$ for which any parametrization $F(t)$, $t \in [a,b]$ has the property that $F(a) = F(b)$, and which does not intersect itself at any other point.

b)



( not simple or closed)

c)



Break $\mathcal{C}$ up into sub-curves which are each the graph of a function of x.

Then $\mathcal{C} = \sum_{i=1}^{n} \mathcal{C}_i$, so

(assume $n=2$ as in picture)

$$\oint_{\mathcal{C}} y\,dx = \int_{\mathcal{C}_1} y\,dx + \int_{\mathcal{C}_2} y\,dx$$

c) contd) parametrize each $\mathcal{C}_i$ by $t=x$.
So here, $\mathcal{C}_1$ is parametrized by $(t, f(t))$, t goes from b to a,
$\mathcal{C}_2$ is parametrized by $(t, g(t))$, t goes from a to b.

$$\int_{\mathcal{C}_1} y\,dx = \int_b^a f(t)\,\frac{dt}{dt}\,dt = \int_b^a f(t)\,dt = -\text{area under } \mathcal{C}_1$$

$$\int_{\mathcal{C}_2} y\,dx = \int_a^b g(t)\,\frac{dt}{dt}\,dt = \int_a^b g(t)\,dt = \text{area under } c_2$$

So $\displaystyle\int_{\mathcal{C}_1} y\,dx + \int_{\mathcal{C}_2} y\,dx = -(\text{area under } c_1 - \text{area under } c_2)$

$= -\text{area between } c_1 \text{ and } c_2$

IV. If $F(t)$ parametrizes $\mathcal{C}$ by arclength, this means $\|F'(t)\| = 1$. So $\|F'(t)\|^2 = 1^2$. So

$$F'(t) \cdot F'(t) = \|F'(t)\|^2 = 1.$$

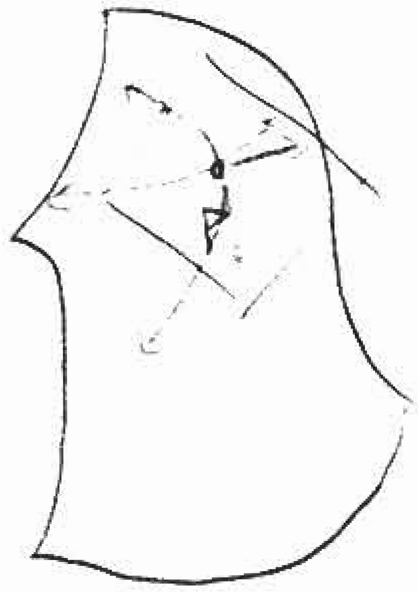
Differentiate both sides:

$F'(t) \cdot F''(t) + F''(t) \cdot F'(t) = 0$ by dot product rule

$\Rightarrow 2F'(t) \cdot F''(t) = 0$ since $\cdot$ is commutative

So $F'(t) \cdot F''(t) = 0$, i.e, $F'(t) \perp F''(t)$.

I. a) The tangent plane to S at p is the plane consisting of all vectors tangent to curves passing through p and lying entirely on S.



b) Tangent plane at $(1,2,1)$ has equation

$$0 = \frac{\partial g}{\partial x}(1,2,1)(x-1) + \frac{\partial g}{\partial y}(1,2,1)(y-2) + \frac{\partial g}{\partial z}(1,2,1)(z-1)$$

where $g(x,y,z) = 2xy - x^2 + \frac{y^2}{z}$

(Since S is a level surface of g)

$\frac{\partial g}{\partial x} = 2y - 2x$ at $(1,2,1) = 2$

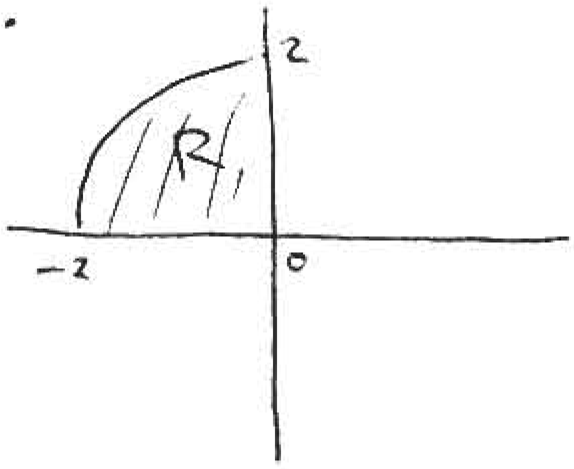
$\frac{\partial g}{\partial y} = 2x + \frac{2y}{z}$ at $(1,2,1) = 6$

$\frac{\partial g}{\partial z} = -\frac{y^2}{z^2}$ at $(1,2,1) = -4$

So the equation is

$$0 = 2(x-1) + 6(y-2) - 4(z-1).$$

VI.



$$0 \le r \le 2 \qquad \sqrt{x^2+y^2} = r$$
$$\frac{\pi}{2} \le \theta \le \pi$$

$$\text{so } \int_R \sqrt{x^2+y^2}\, dA = \int_{\frac{\pi}{2}}^{\pi}\int_0^2 r \; r\,dr\,d\theta = \int_{\frac{\pi}{2}}^{\pi}\int_0^2 r^2\,dr\,d\theta$$

$$= \int_{\frac{\pi}{2}}^{\pi} \left[\frac{r^3}{3}\right]_0^2 d\theta = \int_{\frac{\pi}{2}}^{\pi} \frac{8}{3}\, d\theta = \frac{8}{3}\cdot\frac{\pi}{2} = \frac{4\pi}{3}$$

VII. minimize $x^2+y^2+z^2$ subject to $x+y-z=1$

$g(x,y,z,\lambda) = x^2+y^2+z^2 - \lambda(x+y-z-1)$:

$$\frac{\partial g}{\partial x} = 2x - \lambda = 0 \qquad \lambda = 2x \qquad \text{so } x = y = -z$$

$$\frac{\partial g}{\partial y} = 2y - \lambda = 0 \qquad \lambda = 2y$$

$$\frac{\partial g}{\partial z} = 2z + \lambda = 0 \qquad \lambda = -2z$$

$$\frac{\partial g}{\partial \lambda} = -(x+y-z-1) = 0 \quad \Rightarrow \quad x + x + x - 1 = 0$$

$$\Rightarrow 3x - 1 = 0 \quad \Rightarrow x = \frac{1}{3}$$

so $\left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)$ is the closest point to the origin on the plane.

VIII a) The three line integrals are

$$\int_{\mathcal{C}} f\,ds = \int_a^b f(x(t),y(t))\underbrace{\sqrt{(x'(t))^2+(y'(t))^2}}_{=\frac{ds}{dt}}\,dt$$

s = arclength, $\frac{ds}{dt}$ = speed,

$$\int_{\mathcal{C}} f\,dx = \int_a^b f(x(t),y(t))\;x'(t)\,dt$$

$$\int_{\mathcal{C}} f\,dy = \int_a^b f(x(t),y(t))\;y'(t)\,dt$$

where $(x(t),y(t))$, $t\in[a,b]$ is any parametrization of $\mathcal{C}$

b) $\int_{\mathcal{C}} 2xy^2\,ds$ $\quad\mathcal{C}$ is line $(0,0)$ to $(2,1)$

$(x(t),y(t)) = (2t,t)$ $\quad t\in[0,1]$

$(x'(t),y'(t)) = (2,1)$ $\quad \sqrt{x'(t)^2+y'(t)^2} = \sqrt{5}$

so $$\int_{\mathcal{C}} 2xy^2\,ds = \int_0^1 2(2t)(t^2)\sqrt{5}\,dt = \int_0^1 4\sqrt{5}\,t^3\,dt = \left[\frac{t^4}{4}4\sqrt{5}\right]_0^1$$

$= \sqrt{5}$