PRACTICE MIDTERM #2
SOLUTIONS

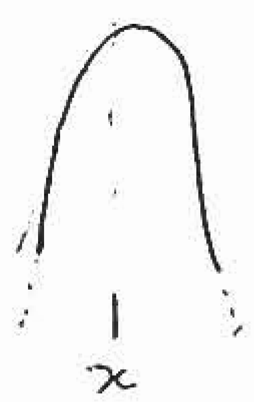
I a) If the function $f(x,t)$ describes the heat of a rod at point x at time t, then, assuming the rod is in a vacuum and homogeneous, $f(x,t)$ will satisfy the equation

$$\frac{\partial^2 f}{\partial x^2} = c\frac{\partial f}{\partial t}$$

called the <u>heat equation</u>, where c is a constant depending on units and conductivity of the rod material.

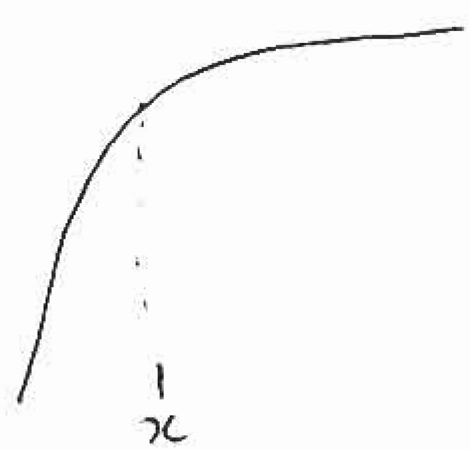
b) We know that heat moves from hotter to colder regions at a rate proportional to the difference in temperatures. This means that if the heat distribution looks like this at a point:

fig 1)



That is, if $\frac{\partial^2 f}{\partial x^2} < 0$ and large, then heat will be moving away from that point quickly in both directions, so the heat at that point will decrease quickly, that is $\frac{\partial f}{\partial t}(x,t) < 0$ and large.

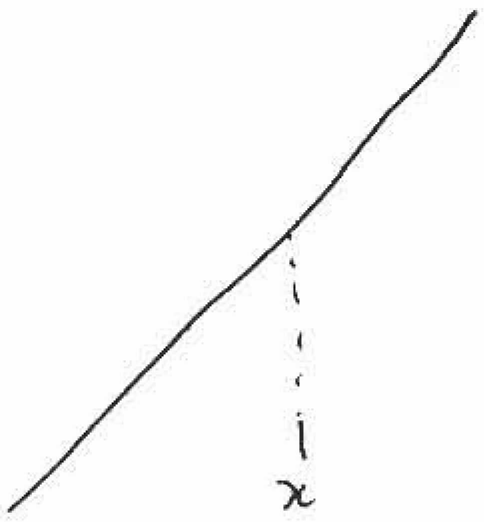
If the distribution looks like this: that is, $\frac{\partial^2 f}{\partial x^2} < 0$ but not so large, then heat will be moving away from x quickly to the left and to the point slowly from the right. So overall, the heat at x will decrease, but not as quickly as in (fig 1)



(fig 2)

ie, $\frac{\partial f}{\partial t}(x,t) < 0$, but not so large

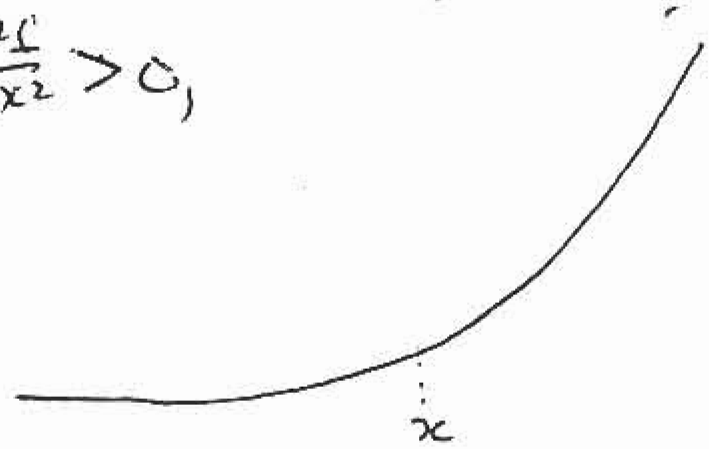
If the distribution looks like this:
that is, $\frac{\partial^2 f}{\partial x^2} = 0$,



(fig 3)

then heat is moving to x from the right and from x to the left at equal rates, so the heat at x will be constant, ie, $\frac{\partial f}{\partial t}(x,t) = 0$

If the distribution looks like:
that is, $\frac{\partial^2 f}{\partial x^2} > 0$,



(fig 4)

then heat is moving quickly to x from the right and slowly from x to the left, so the heat at x will be increasing, ie, $\frac{\partial f}{\partial t}(x,t) > 0$

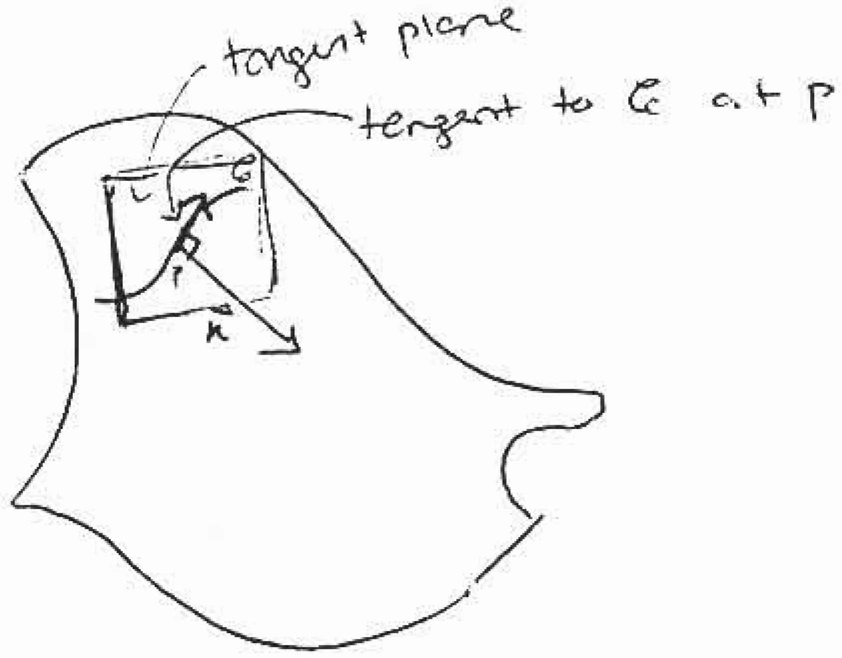
And, finally, if the distribution looks like:
(that is, $\frac{\partial^2 f}{\partial x^2} > 0$ and large,)

(fig 5)

then heat is moving to x quickly from both sides, so the heat at x is increasing quickly, ie, $\frac{\partial f}{\partial t} > 0$ and large.

So we notice that the relationship between $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial f}{\partial t}$ is exactly that captured by the heat equation, $\frac{\partial^2 f}{\partial x^2} = c\frac{\partial f}{\partial t}$.

II a) <u>Def</u> A vector $\vec{n}$ is said to be <u>normal</u> to a surface S at a point p if it is perpendicular to the tangent plane to S at p, that is, if it is perpendicular to the tangent vector at p to any curve $\mathcal{C}$ lying entirely on the surface S and passing through p.



b) Let $\mathcal{C}$ be a curve contained on the level surface of $f(x,y,z)$ which passes through p. Then if $F(t) = (x(t), y(t), z(t))$ parametrizes $\mathcal{C}$, $F'(t)$ is the tangent vector to $\mathcal{C}$ at $F(t)$.

So we need to show $\nabla f \perp F'(t)$, ie $\nabla f \cdot F'(t) = 0$.

But if $u = f \circ F(t) = f(x(t), y(t), z(t))$ then by construction, since $\mathcal{C}$ lies on a level surface $f(x,y,z) = c$, u is just the constant function $u(t) = c$,

So $\frac{du}{dt} = 0$. But by the chain rule, we also have

$\frac{du}{dt} = \nabla f \cdot F'(t)$, so $\nabla f \cdot F'(t) = 0$ as required.

III. Estimate the % error made in calculating the kinetic energy, $E_k(m,v) = \frac{1}{2}mv^2$, of a car if your mass measurement is accurate to within 5% and your velocity measurement is accurate to within 2%.

$$\% \text{ error} = \left|\frac{dE_k}{E_k}\right| \leq \frac{\left|\frac{\partial E_k}{\partial m}\Delta m\right| + \left|\frac{\partial E_k}{\partial v}\Delta v\right|}{|E_k(m,v)|}$$

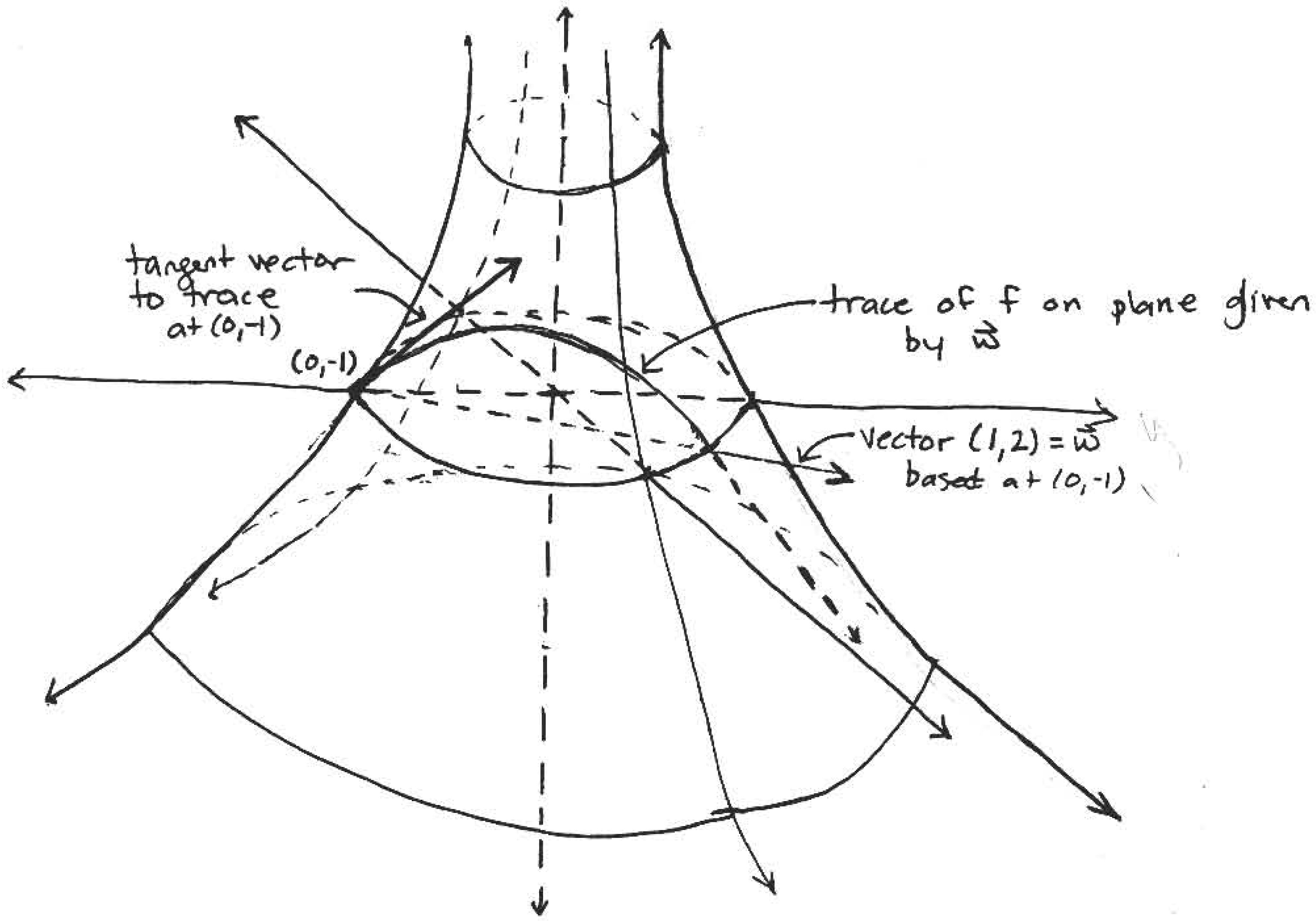
$$= \frac{\left|\frac{v^2}{2}\cdot\Delta m\right| + |mv\cdot\Delta v|}{|\frac{1}{2}mv^2|}$$

$$= \frac{\left|\frac{v^2}{2}\ .05m\right| + |mv\cdot .02v|}{\frac{1}{2}mv^2}$$

$$= \frac{\left|.05\left(\frac{1}{2}mv^2\right)\right| + \left|.04\left(\frac{1}{2}mv^2\right)\right|}{\frac{1}{2}mv^2} = .09$$

So 9% error is made in the Energy estimate.

IV a) This is a rotational surface:



b) $D_{\vec{u}} f(0,-1) = \nabla f(0,-1) \cdot \vec{u}$

$\vec{u} = \dfrac{\vec{w}}{\|\vec{w}\|} = \dfrac{(1,2)}{\sqrt{5}}$

$\nabla f(x,y) = \left(\dfrac{-2x}{x^2+y^2}, \dfrac{-2y}{x^2+y^2} \right)$

$\nabla f(0,-1) = (0,2)$

$D_{\vec{u}} f(0,-1) = \dfrac{(0,2)\cdot(1,2)}{\sqrt{5}} = \dfrac{4}{\sqrt{5}}$

This is the slope of the tangent vector at (0,1) to the trace of the graph on the plane determined by $\vec{w}$.

V. We want to maximize and minimize the function $T(x,y,z) = xyz$ subject to the constraint $x^2+y^2+z^2-1=0$

So we form $L(x,y,z,\lambda) = T(x,y,z) - \lambda(x^2+y^2+z^2-1)$

and find $\frac{\partial L}{\partial x} = yz - 2\lambda x$

$$\frac{\partial L}{\partial y} = xz - 2\lambda y$$

$$\frac{\partial L}{\partial z} = xy - 2\lambda z$$

$$\frac{\partial L}{\partial \lambda} = -x^2-y^2-z^2+1$$

We need to find where these partials are simulteneously equal to zero. We can get both positive and negative values for T on the sphere, so we can assume $x,y,z \neq 0$. (Since $T(0,y,z)=0=T(x,0,z)=T(x,y,0)$ is neither a minimum nor a maximum).

So
$$0 = yz - 2\lambda x \Rightarrow \lambda = \frac{yz}{2x}$$
$$0 = xz - 2\lambda y \Rightarrow \lambda = \frac{xz}{2y}$$
$$0 = xy - 2\lambda z \Rightarrow \lambda = \frac{xy}{2z}$$

So $\frac{y\cancel{z}}{\cancel{2}x} = \frac{x\cancel{z}}{\cancel{2}y} \Rightarrow \frac{y}{x} = \frac{x}{y} \Rightarrow x^2=y^2 \Rightarrow x = \pm y$

Similarly, $y = \pm z$, $x = \pm z$.

So $x^2+y^2+z^2 = x^2+x^2+x^2 = 3x^2$

$3x^2 = 1$ where $x = \pm\frac{1}{\sqrt{3}}$

So the maxima occur where two of x,y,z are $\frac{-1}{\sqrt{3}}$ and one is $\frac{1}{\sqrt{3}}$ or where all are $\frac{1}{\sqrt{3}}$, minima

V cont'd) occur at the other cases

($x = \pm\frac{1}{\sqrt{3}}$, $y = \frac{\pm 1}{\sqrt{3}}$, $z = \frac{\pm 1}{\sqrt{3}}$ are all cases)

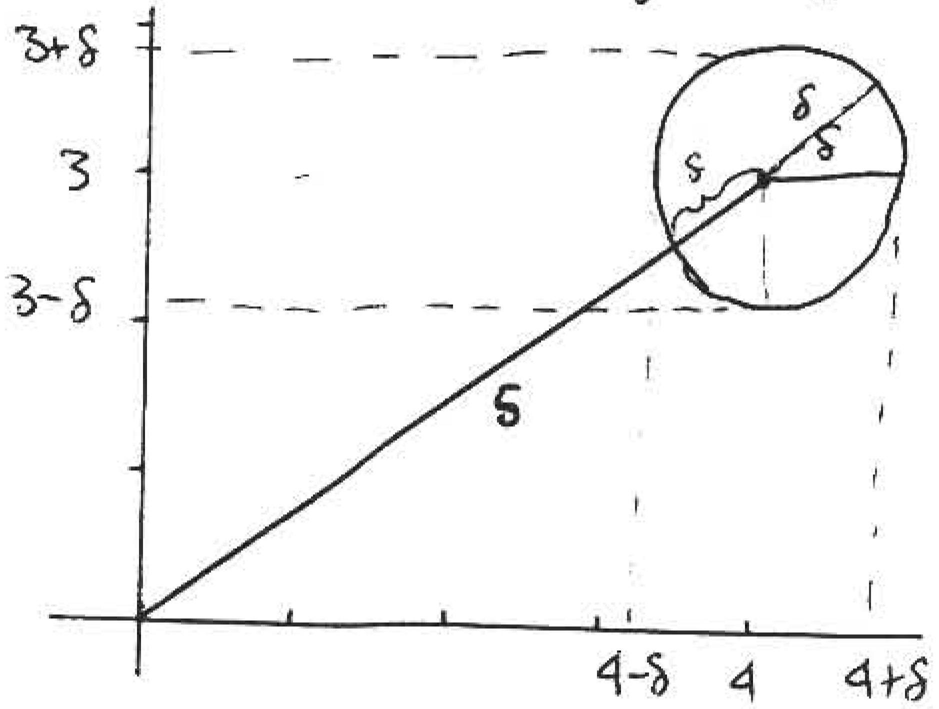
The maximum temperature is $T(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}) = \frac{1}{3}$

The minimum temperature is $T(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}) = \frac{-1}{3}$

VI. a) $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ if for all $\varepsilon > 0$ there is a $\delta > 0$ so if $0 < \|(x,y)-(a,b)\| < \delta$, then $|f(x,y) - L| < \varepsilon$.

b) So given an arbitrary $\varepsilon > 0$, we need to produce a $\delta > 0$ so that if

$0 < \|(x,y) - (4,5)\| < \delta$, then $|x+y+\sqrt{x^2+y^2} - 12| < \varepsilon$.



If $0 < \|(x,y) - (4,3)\| < \delta$, then

$4 - \delta < x < 4 + \delta$ and

$3 - \delta < y < 3 + \delta$.

Also, since $\sqrt{x^2+y^2}$ = distance of (x,y) to $(0,0)$

$$5 - \delta < \sqrt{x^2+y^2} < 5 + \delta$$

So adding these together gives

$$12 - 3\delta < x + y + \sqrt{x^2+y^2} < 12 + 3\delta, \text{ so}$$

$|x+y+\sqrt{x^2+y^2} - 12| < 3\delta$. So if $3\delta = \varepsilon$,

$|x+y+\sqrt{x^2+y^2} - 12| < \varepsilon$ as required. So $\delta = \frac{\varepsilon}{3}$ is the necessary δ.