SOLUTIONS TO PRACTICE MIDTERM #1

I. a) If $\vec{v} = (v_1, v_2, v_3)$, $\vec{w} = (w_1, w_2, w_3)$, then

$$\vec{v} \times \vec{w} = \left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}, - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} \right)$$

b)

$$\vec{v} \cdot (\vec{v} \times \vec{w}) = v_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - v_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + v_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= \begin{vmatrix} v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0$$ since two rows are identical

$$\vec{w} \cdot (\vec{v} \cdot \vec{w}) = w_1 \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - w_2 \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + w_3 \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

$$= \begin{vmatrix} w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0$$ since two rows are identical.

So since $\vec{v} \cdot (\vec{v} \times \vec{w}) = 0$, $\vec{v} \perp (\vec{v} \times \vec{w})$ and since $\vec{w} \cdot (\vec{v} \times \vec{w}) = 0$, then $\vec{w} \perp (\vec{v} \times \vec{w})$.

c) $\vec{v} \times \vec{w}$ is perpendicular to $\vec{v}$ and $\vec{w}$ in the direction which makes $\vec{v}, \vec{w}, \vec{v} \times \vec{w}$ a right-handed system of vectors. Its length is the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.

d)

$$\vec{v} \times \vec{w} = \left(\begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix}, - \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} \right) = (2, 5, -4)$$

III a) Volume $= \left| \det \text{matrix} \right|$

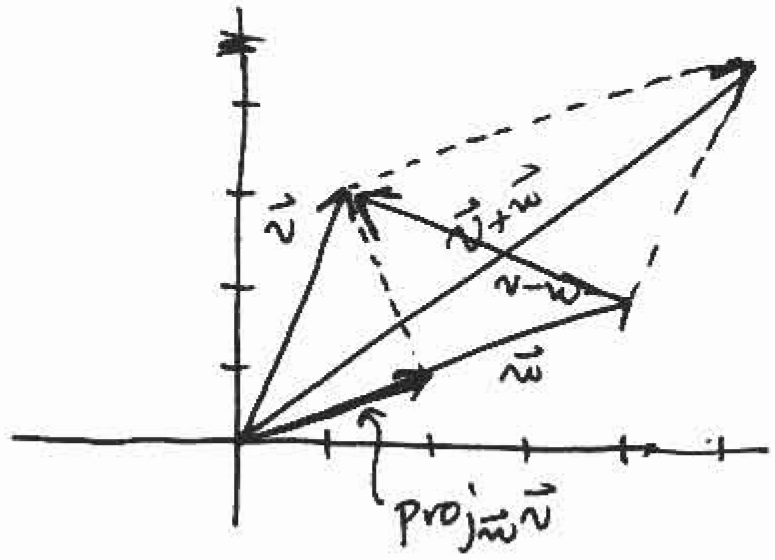
$$= \left| \begin{vmatrix} 1, & 1, & -2 \\ 0, & 3, & 1 \\ -4, & 2, & 1 \end{vmatrix} \right| = \left| 1 \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ -4 & 1 \end{vmatrix} + -2 \begin{vmatrix} 0 & 3 \\ -4 & 2 \end{vmatrix} \right|$$

$$= \left| 1 - 4 + 24 \right| = 21$$

b) distance $= \dfrac{|Ax + By + C|}{\sqrt{A^2+B^2}} = \dfrac{|6 \cdot 2 - 8 \cdot 1 + 3|}{\sqrt{6^2+8^2}} = \dfrac{|12 - 8 + 3|}{\sqrt{36+64}}$

$$= \frac{7}{10}$$

II. a)



b) $\vec{v}+\vec{w}=(5,5)$

$\vec{v}-\vec{w}=(-3,1)$

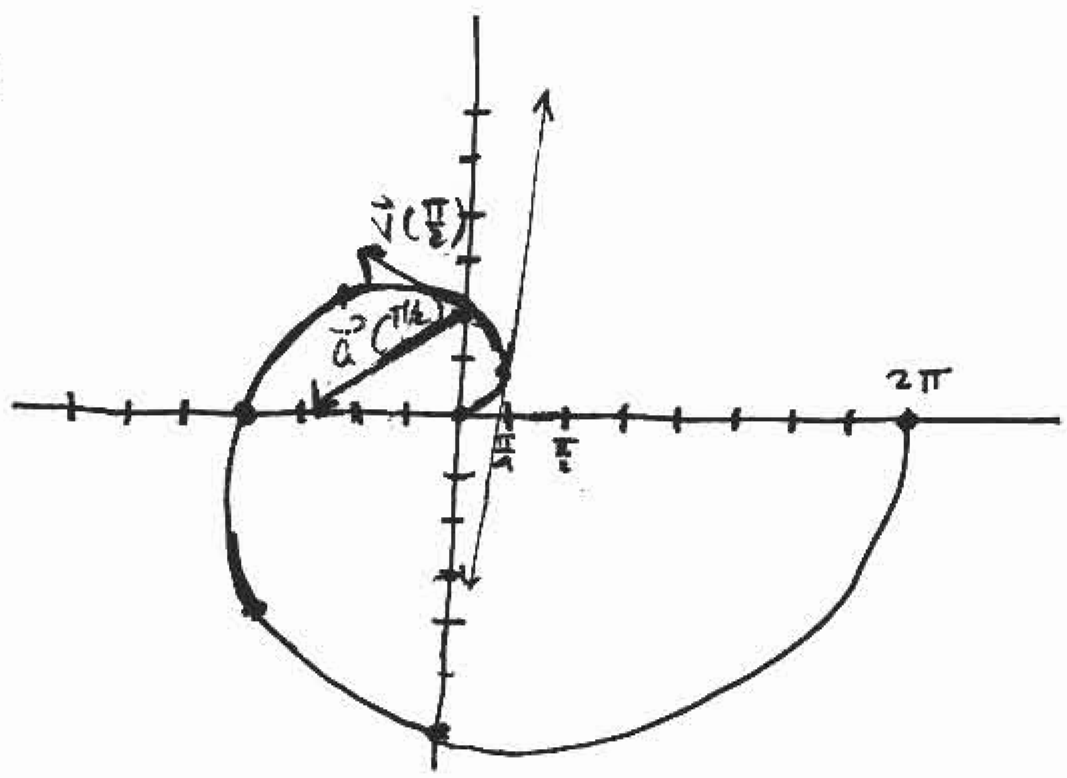
c) $proj_{\vec{w}}\vec{v} = \frac{\vec{v}\cdot\vec{w}}{\|w\|^2}\vec{w}$

$= \frac{(1\cdot 4+3\cdot 2)}{4^2+2^2}(4,2)$

$= \frac{10}{20}(4,2) = (2,1)$

IV. a) Def We say a curve $\mathcal{C}$ is parametrized by a vector valued function $F:\mathbb{R}\to\mathbb{R}^n$ if $\mathcal{C}=\{F(t)\mid t\in\mathbb{R}\}$. Then we call F a Parametrization of $\mathcal{C}$.

b)



c) $\text{arclength} = \int_0^{2\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$

$\frac{dx}{dt} = \cos t + -t\sin t \qquad \left(\frac{dx}{dt}\right)^2 = \cos^2 t - 2t\cos t\sin t + t^2\sin^2 t$

$\frac{dy}{dt} = \sin t + t\cos t \qquad \left(\frac{dy}{dt}\right)^2 = \sin^2 t + 2t\cos t\sin t + t^2\cos^2 t$

$\text{arclength} = \int_0^{2\pi}\sqrt{\cos^2 t - \cancel{2t\cos t\sin t} + t^2\sin^2 t + \sin^2 t + \cancel{2t\cos t\sin t} + t^2\cos^2 t}\,dt$

III c) cont'd)

$$= \int_0^{2\pi} \sqrt{\cos^2 t + \sin^2 t + 2t(\cos^2 t + \sin^2 t)}\, dt$$

$$= \int_0^{2\pi} \sqrt{1+2t}\, dt$$

let $u = 1+2t$ $\quad \int \sqrt{1+2t}\,dt = \frac{1}{2}\int \sqrt{u}\, du = \frac{2}{3}\cdot\frac{1}{2} u^{3/2} + c$

$du = 2dt$

$$= \frac{1}{3}(1+2t)^{3/2} + c$$

so $\displaystyle\int_0^{2\pi} \sqrt{1+2t}\, dt = \left[\frac{1}{3}(1+2t)^{3/2}\right]_0^{2\pi} = \frac{1}{3}(1+4\pi)^{3/2}$

d) Velocity is $\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (\cos t - t\sin t,\ \sin t + t\cos t)$

at $t = \pi/2$, this $= \left(-\frac{\pi}{2}, 1\right)$

e) $\|\vec{v}(t)\| = \sqrt{1+2t}$ (from part c)

$\to \infty$ as $t \to \infty$

So the particle is speeding up

f) $\vec{a}(t) = \vec{v}\,'(t) = (-\sin t - \sin t - t\cos t,\ \cos t + t\cos t - t\sin t)$

$= (-2\sin t - t\cos t,\ (1+t)\cos t - t\sin t)$

at $t = \frac{\pi}{2}$ $\quad = \left(-2, -\frac{\pi}{2}\right)$

g) parametric equation is $(x,y) = \underset{\text{point}}{(x_0, y_0)} + t\underset{\text{vector}}{(\vec{v}_1, \vec{v}_2)}$

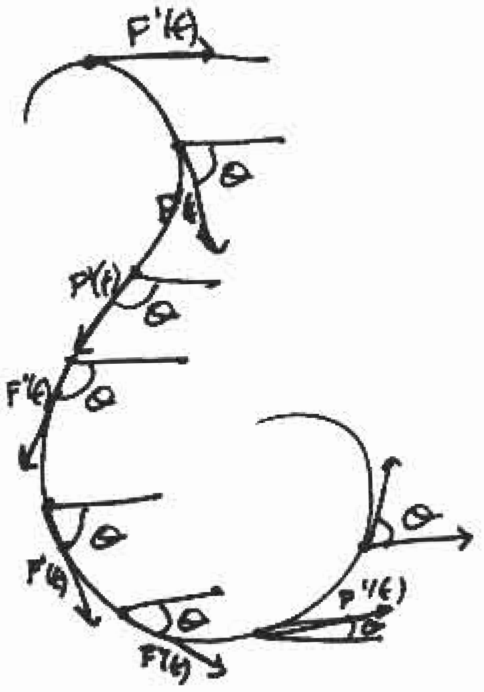
$$= \left(\frac{\pi}{4}\cos\frac{\pi}{4}, \frac{\pi}{4}\sin\frac{\pi}{4}\right) + t\left(\cos\frac{\pi}{4} - \frac{\pi}{4}\sin\frac{\pi}{4},\ \sin\frac{\pi}{4} + \frac{\pi}{4}\cos\frac{\pi}{4}\right)$$

$$= \left(\frac{\pi\sqrt{2}}{8}, \frac{\pi\sqrt{2}}{8}\right) + t\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}\pi}{8},\ \frac{\sqrt{2}}{2} + \frac{\sqrt{2}\pi}{8}\right)$$

V. a) <u>Def</u> Let $F(t)$ parametrize $\mathcal{C}$ by arclength. Let $\theta(t)$ denote the angle between $F'(t)$ and $(1,0)$. Then the curvature of $\mathcal{C}$ at $F(t)$ is

$$K(t) = \left| \frac{d\theta(t)}{dt} \right|$$

b)



c)

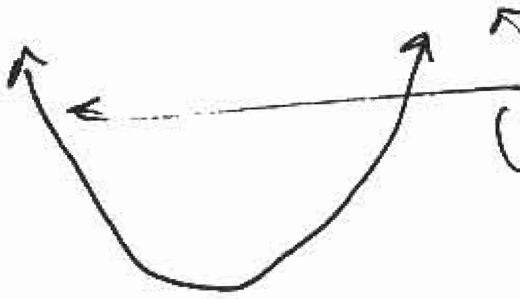
$$k = \frac{\left| d^2y/dx^2 \right|}{\left| 1 + \left(dy/dx \right)^2 \right|^{3/2}} =$$

$x = t$ so $y = x^2$

$$\frac{dy}{dx} = 2x \qquad \frac{d^2y}{dx^2} = 2$$

$$= \frac{|2|}{\left|1+(2x)^2\right|^{3/2}} = \frac{2}{\left|1+4x^2\right|^{3/2}}$$

d) as $x \to \pm\infty$ $(t \to \pm\infty)$, $k \to 0$, so the curve becomes more and more straight as $t \to \pm\infty$



↖ becomes more straight
(it is a parabola)