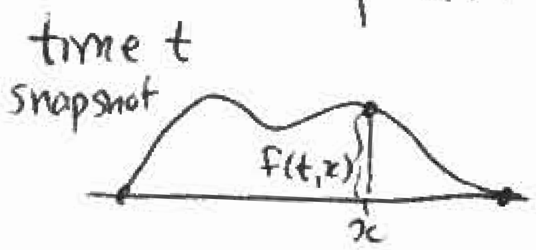
MIDTERM #2

HONOR PLEDGE ____________ NAME ________ KEY

NO AIDS such as calculators, books or notes allowed on this exam. There are a total of 105 points possible. I will grade out of 100.

I. a) State the wave equation. If $f(t,x)$ describes the height at time t of an (ideal) wave over the point x, then f satisfies the differential equation

(10)



$$\frac{\partial^2 f}{\partial t^2} = k^2 \frac{\partial^2 f}{\partial x^2}$$ for some constant k, which depends on units and properties of the string.

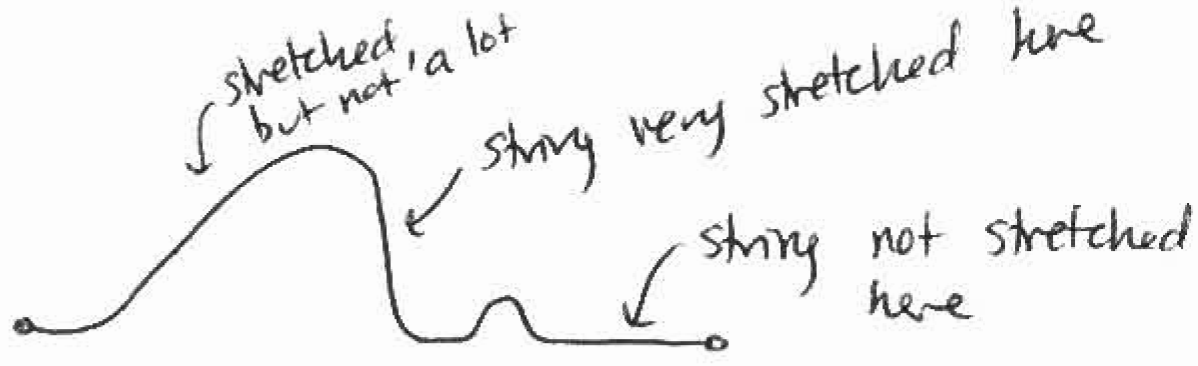
b) Essay: Explain why the wave equation is the correct equation to describe a vibrating string.

(15)

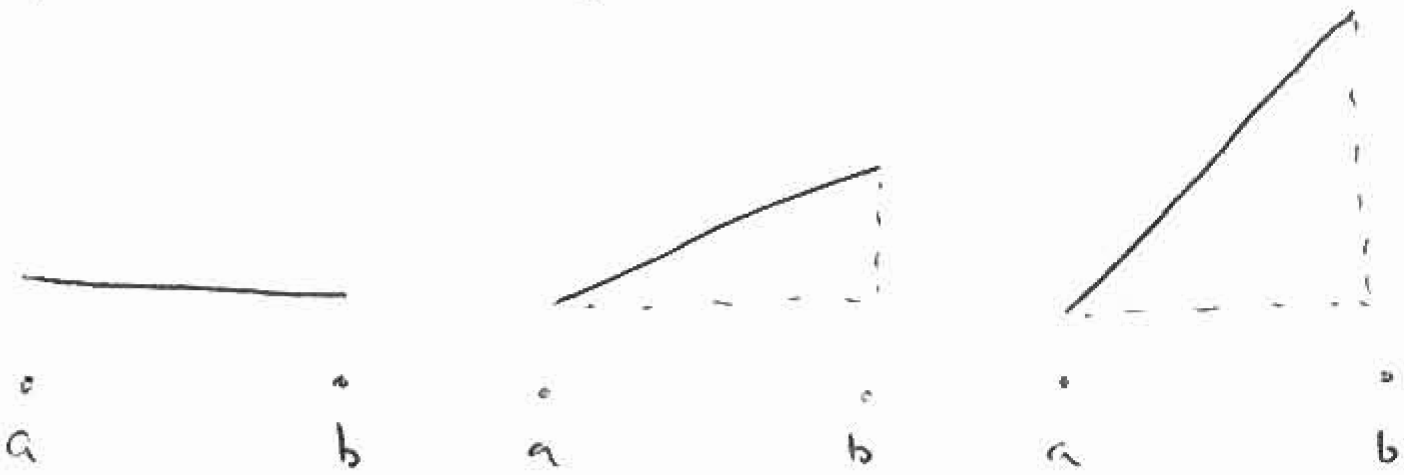
The second derivative of f with respect to x is the curvature of the wave over x at time t, or the change in tangent slopes at that point and time. The second derivative of f with respect to t is the acceleration of the wave. Since force is proportional to acceleration ($F=ma$), This equation says that the force in a wave type system (tension) is proportional to the curvature of the wave.

Why is this the correct relationship? The restoring force of a stretched string (tension) is proportional to how stretched out it is.

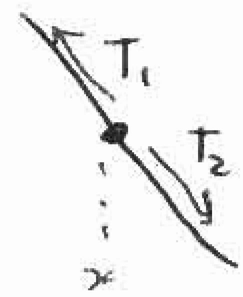
Consider a wave:



How stretched the string is over a given length [a,b] is related to the average slope of the string over that length:



As the slope increases, the distance that piece of string has to stretch increases.

So on a piece of the wave shaped like this: 

the two tension forces are equal & opposite, so the overall force on that point = 0.

on a piece shaped like 

, there is a cumulative downward force

and so on. So a cumulative force on the wave around x at time t occurs when the slopes are changing at x. Ie., when the second derivative in $x > 0$, so is the force. When it is < 0, the force points the opposite way.

II a) Define directional derivative.

(10) If $\vec{u} = (u_1, u_2)$ is a unit vector and $f(x,y)$ is defined near (a,b), then the directional derivative is

$$D_{\vec{u}} f(a,b) = \lim_{t \to 0} \frac{f((a,b) + t\vec{u}) - f(a,b)}{t}$$

if this limit exists.

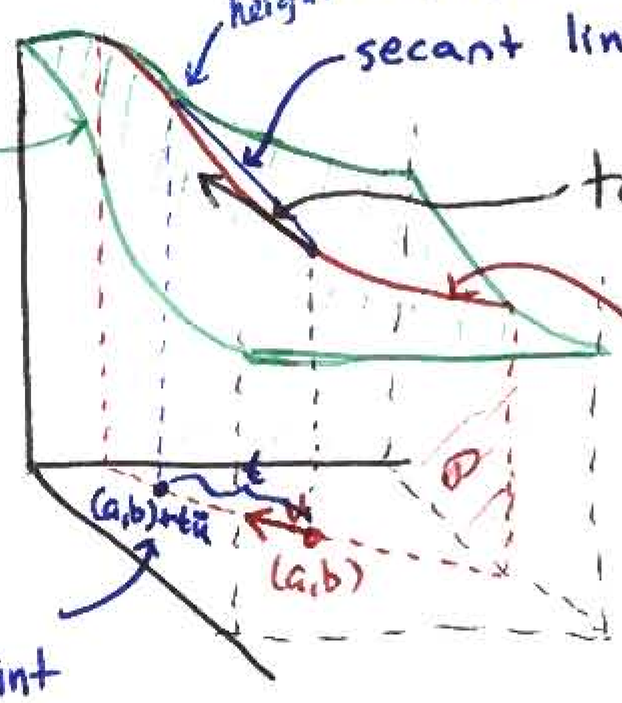
b) Draw a picture and use it to explain what this means and why this definition is the

(10) right one to describe that.

height is $f((a,b)+t\vec{u})$

secant line through $(a,b,f(a,b))$ and $(a+tu_1, b+tu_2, f((a,b)+t\vec{u}))$

graph of $f(x,y)$



tangent vector to trace of graph of f on plane $\perp$ (xy plane) and containing $\vec{u}$ (slope of this vector $= D_{\vec{u}} f(a,b)$)

trace of graph of f on plane $P \perp$ (xy plane) and containing $\vec{u}$ (or, P = plane given by $\vec{u}$ and $(0,0,1)$).

$(a,b)+t\vec{u}$ is the point at distance t from (a,b) along $\vec{u}$ direction.

Since $\dfrac{f((a,b)+t\vec{u}) - f(a,b)}{t} = \dfrac{\text{rise}}{\text{run}}$ is the slope of the secant line through $(a,b,f(a,b))$ and the point $(a+tu_1, b+tu_2, f((a,b)+t\vec{u}))$, as $t \to 0$, the limit of those slopes is the slope of the tangent vector to the graph in the $\vec{u}$ direction at $(a,b,f(a,b))$,

State and
c) prove the directional derivative theorem.

(15)

<u>Theorem</u> If $\vec{u}$ is a unit vector and the function $f(x,y)$ has continuous partial derivatives near (a,b), then $D_{\vec{u}}f(a,b) = \nabla f(a,b) \cdot \vec{u}$

<u>proof</u>

If $\vec{u}$ is a unit vector, then by definition,

$$D_{\vec{u}} f(a,b) = \lim_{t\to 0} \frac{f((a,b)+t\vec{u}) - f(a,b)}{t} = \lim_{t\to 0} \frac{f(a+tu_1, b+tu_2) - f(a,b)}{t}.$$

Let $g(t) = f(x(t), y(t))$ where $x(t) = a+tu_1$, $y(t) = b+tu_2$.

Then the limit above can be rewritten as

$D_{\vec{u}} f(a,b) = \lim_{t\to 0} \frac{g(t) - g(0)}{t}$, which by definition of the derivative of a function of one variable, is just $g'(0)$. However, since f, by assumtion, has continuous partials near $(a,b) = (x(0), y(0))$, and since $x(t)$ and $y(t)$ are differentiable functions of t, we can apply the chain rule to g to find

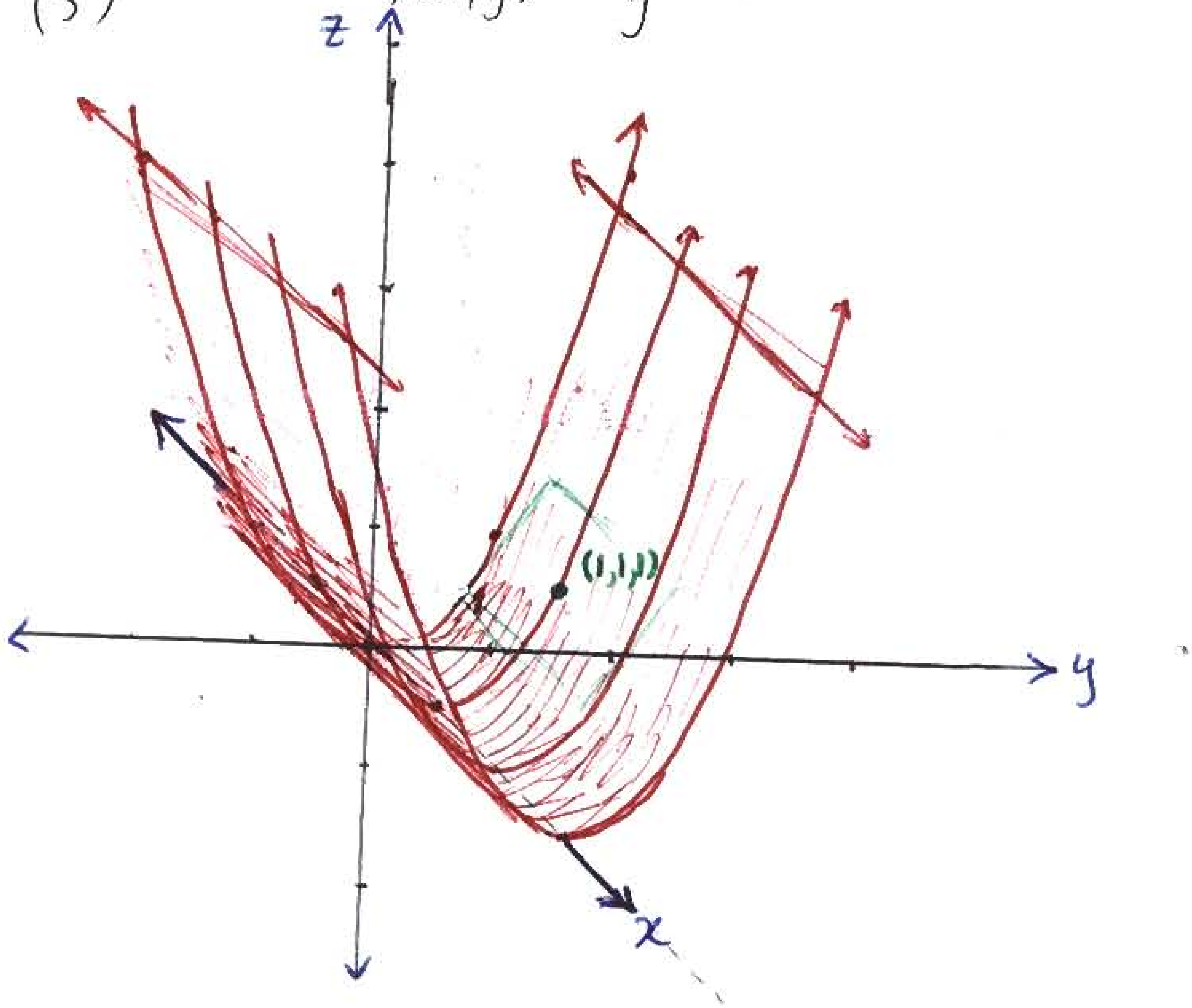
$$D_{\vec{u}} f(a,b) = \frac{dg}{dt}(0) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \frac{\partial f}{\partial x}(a,b)\,(u_1) + \frac{\partial f}{\partial y}(a,b)\,(u_2)$$

$$= \nabla f(a,b) \cdot \vec{u},$$

and we are done ✓.

III a) Draw the graph of the function

(5) $f(x,y) = y^2$



b) Find the equation of the tangent plane to the surface in a) at the point $(1,1,1)$.

(10)

The tangent plane at $(a,b,f(a,b))$ to the graph of a function $f(x,y)$ is given by

$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$ so for $f(x,y) = y^2$,

$\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 2y$. So at $(1,1,1)$,

$z = 1 + 0(x-1) + 2(y-1) = 1 + 2(y-1) = 2y - 1$

IV. Suppose you are determining the gravitational constant, g, at the earth's surface by dropping a penny

(10) from the top of a staircase and timing how long it takes to hit the ground. Neglecting wind resistence, Assume that the height of the penny satisfies

$h(t) = -\frac{g}{2}t^2 + h_0$ as it drops, so

$g = \frac{2h_0}{t^2}$, where h_0 is the height of the stairs and t^2 is the time measured.

If after several trials, you determine a value for h_0 up to an error of 2% and a value for t up to an error of 5%. With what % error can you determine g?

$$\% \text{ error} = \frac{|dg|}{|g|} \leq \frac{\left|\frac{\partial g}{\partial h_0}(h_0,t)(.02h_0)\right| + \left|\frac{\partial g}{\partial t}(h_0,t)(.05t)\right|}{\left|\frac{2h_0}{t^2}\right|}$$

$$= \frac{\left|\frac{2}{t^2}(.02h_0)\right| + \left|-\frac{4h_0}{t^{\cancel{3}2}}(.05\cancel{t})\right|}{\left|\frac{2h_0}{t^2}\right|}$$

$$= \frac{\left|\frac{2\cancel{h_0}}{\cancel{t^2}}(.02)\right| + \left|-2\left(\frac{2\cancel{h_0}}{\cancel{t^2}}\right)(.05)\right|}{\cancel{\left|\frac{2h_0}{t^2}\right|}}$$

$$= .02 + 2(.05) = .12$$

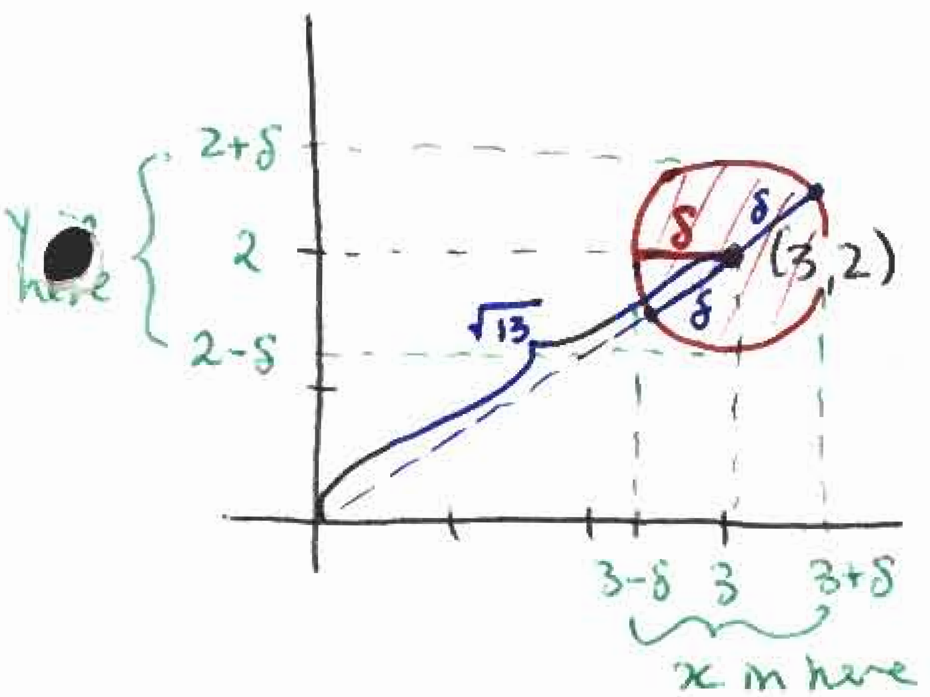
12% error results in g estimate.

I. Use the definition of limit to prove that

(10) $$\lim_{(x,y)\to(3,2)} 2x - y - \sqrt{x^2+y^2} = 4 - \sqrt{13}$$

To show this, we need to find for any $\varepsilon > 0$ a δ (in terms of ε) with the property that if $0 < \|(x,y) - (3,2)\| < \delta$, then $|(2x - y - \sqrt{x^2+y^2}) - (4 - \sqrt{13})| < \varepsilon$.

So suppose we know the first inequality is true.



That means (x,y) is somewhere in the δ radius disk around $(3,2)$. If this is true, we get the following bounds:

$$3 - \delta < x < 3 + \delta$$
$$2 - \delta < y < 2 + \delta$$
$$\sqrt{13} - \delta < \sqrt{x^2+y^2} < \sqrt{13} + \delta$$

We want to combine these inequalities to get a bound on $2x - y - \sqrt{x^2+y^2}$, so multiply the first one by 2, and the second & third by -1:

$$\left.\begin{array}{l} 2(3-\delta) < 2x < 2(3+\delta) \\ -(2-\delta) > -y > -(2+\delta) \\ -(\sqrt{13}-\delta) > -\sqrt{x^2+y^2} > -(\sqrt{13}+\delta) \end{array}\right\} \Rightarrow \begin{array}{l} 6 - 2\delta < 2x < 6 + 2\delta \\ -2 - \delta < -y < -2 + \delta \\ -\sqrt{13} - \delta < -\sqrt{x^2+y^2} < -\sqrt{13} + \delta \end{array}$$

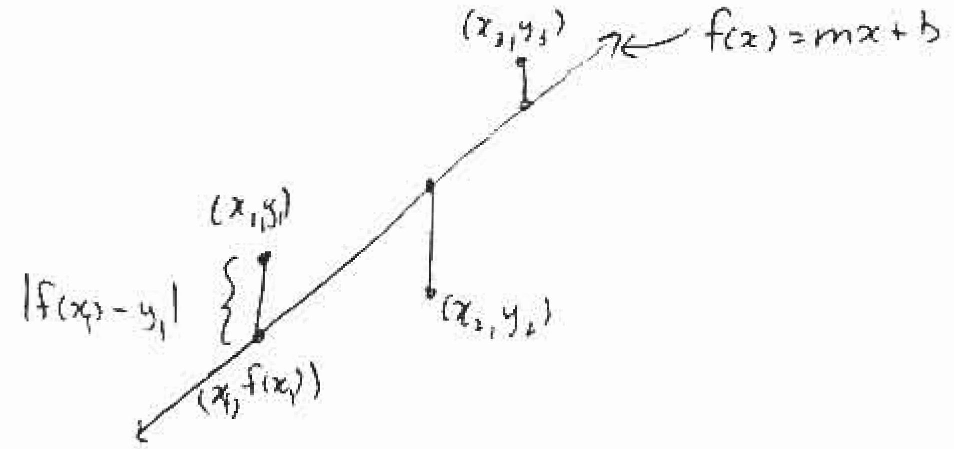
and add

$$4 - \sqrt{13} - 4\delta < 2x - y - \sqrt{x^2+y^2} < 4 - \sqrt{13} + 4\delta$$

This is equivalent to $|(2x - y - \sqrt{x^2+y^2}) - (4 - \sqrt{13})| < 4\delta$, so if we let $\delta = \frac{\varepsilon}{4}$, then we get $|(2x - y - \sqrt{x^2+y^2}) - (4 - \sqrt{13})| < \varepsilon$ as

VI. Find the best straight-line fit for the three data points $(1,4)$, $(3,6)$, $(5,11)$ by minimizing (in m, b) the sums of $(f(x_i)-y_i)^2$ for these three points, assuming $f(x) = mx + b$.

(10)



[let $L(m,b) = \sum_{i=1}^{3} (mx_i + b - y_i)^2$, and minimize this with respect to m and b.]

$$L(m,b) = (mx_1 + b - y_1)^2 + (mx_2 + b - y_2)^2 + (mx_3 + b - y_3)^2$$

$$= (m+b-4)^2 + (3m+b-6)^2 + (5m+b-11)^2$$

To minimize, find critical point:

$$\frac{\partial L}{\partial m} = 2(m+b-4)\cdot 1 + 2(3m+b-6)\cdot 3 + 2(5m+b-11)\cdot 5 = 0$$

$$\frac{\partial L}{\partial b} = 2(m+b-4) + 2(3m+b-6) + 2(5m+b-11) = 0$$

$$\Rightarrow \begin{cases} \cancel{2}(35m + 9b - 77) = 0 \\ \cancel{2}(\underset{3}{\cancel{9}}m + \cancel{3}b - \underset{7}{\cancel{21}}) = 0 \end{cases} \Rightarrow \quad 3m + b - 7 = 0 \Rightarrow b = 7 - 3m$$

plug into 1st eqn: $35m + 9(7-3m) - 77 = 0$

$$35m + 63 - 27m - 77 = 0$$

$$8m - 14 = 0 \Rightarrow m = \frac{14}{8} = \frac{7}{4}$$

and plug m back in for b: $b = 7 - 3\left(\frac{7}{4}\right) = \frac{28}{4} - \frac{21}{4} = \frac{7}{4}$

So the best straight line fit is $\boxed{y = \frac{7}{4}x + \frac{7}{4}}$.

(Clearly there is no maximum, as by taking b larger, we get larger error, not so easy to justify this point is not a saddle [illegible]