MIDTERM #1

HONOR PLEDGE ______________________ NAME ______ KEY ______

- No calculators, books, or notes allowed on this exam. There are 105 points possible. I will grade out of 100.

(5 pts) I a) Define curve associated to a vector-valued function

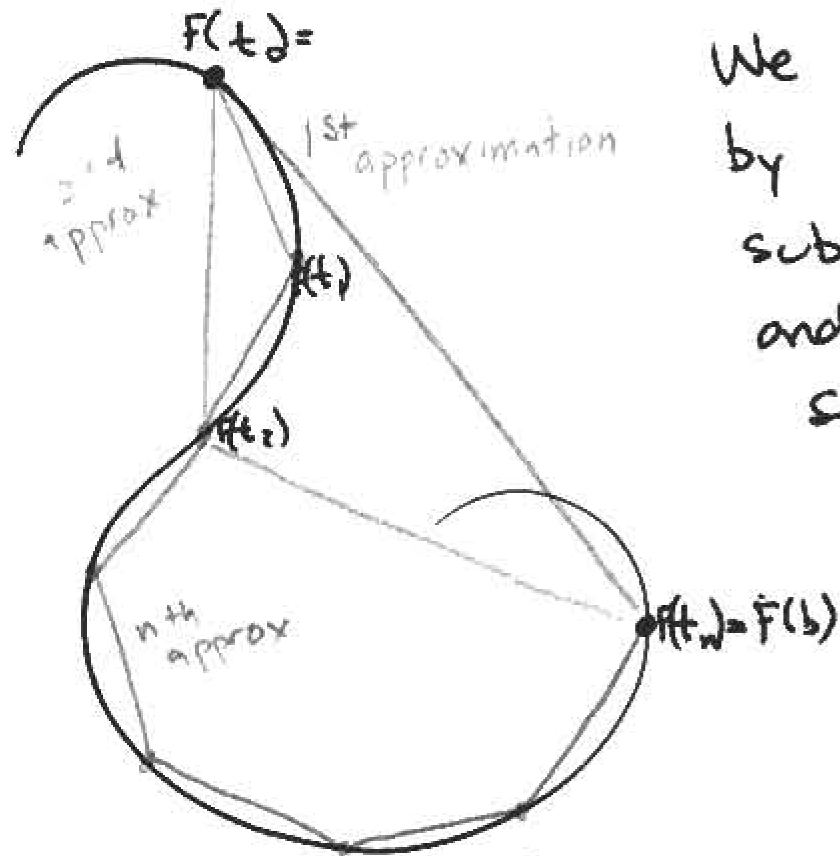
The curve associated to $F:\mathbb{R}\to\mathbb{R}^n$ is the set of points $\mathcal{C}=\{F(t) \mid t\in\mathbb{R}\}\subset\mathbb{R}^n$.

(5 pts) b) State the arclength theorem.

If $F(t)=(x_1(t),\dots,x_n(t))$ parametrizes $\mathcal{C}$ between $F(a)$ and $F(b)$ and if $F'(t)$ exists and is continuous on $[a,b]$, then the arclength of $\mathcal{C}$ between $F(a)$ and $F(b)$ is given by the formula

$$\text{arclength}=\int_a^b\sqrt{\left(\frac{\partial x_1}{\partial t}\right)^2+\dots+\left(\frac{\partial x_n}{\partial t}\right)^2}\,dt$$

(10 pts) c) How did we prove this? Draw a picture and use it to explain the idea of the proof.



We approximated the arclength $F(a)$ to $F(b)$ (in $\mathbb{R}^2$) by breaking up the interval $[a,b]$ into subintervals given by $t_0=a<t_1<\dots<t_n=b$ and adding up the lengths of line segments between $F(t_{i-1})$ and $F(t_i)$.

This gave

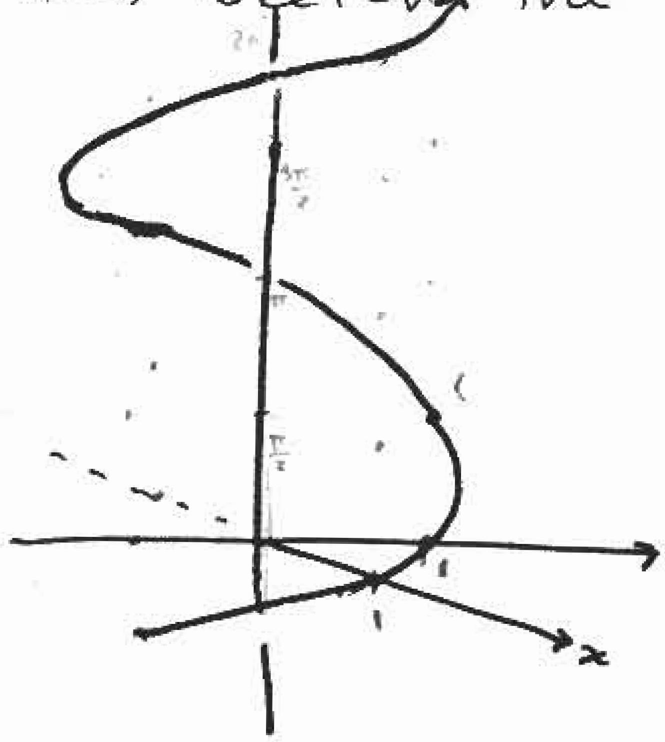
$$\sum_{i=1}^{n}\|F(t_i)-F(t_{i-1})\|=\sum_{i=1}^{n}\sqrt{(x_1(t_i)-x_1(t_{i-1}))^2+\dots+(x_n(t_i)-x_n(t_{i-1}))^2}$$

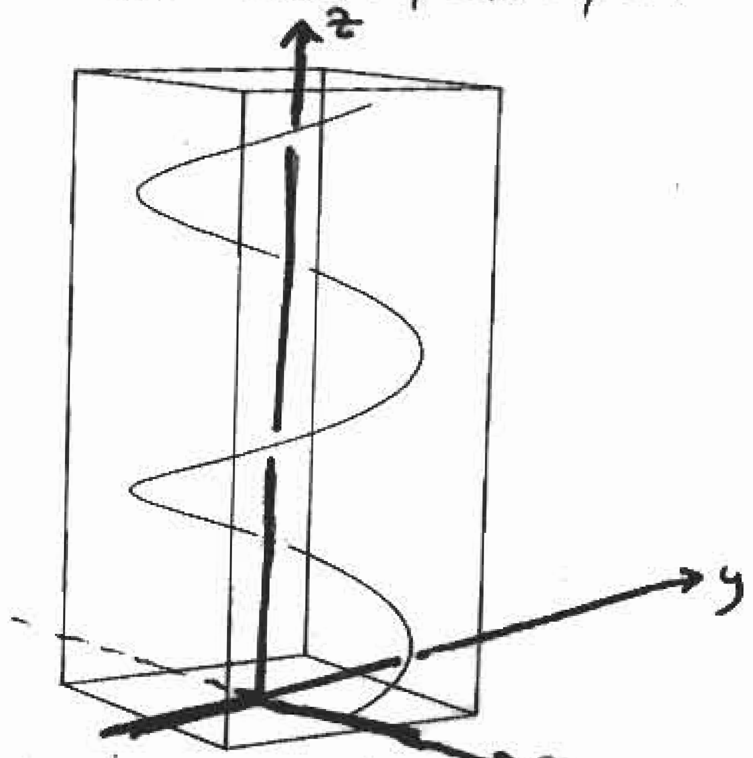
Then we multiplied by $\frac{t_i-t_{i-1}}{t_i-t_{i-1}}$ to get

$$\sum_{i=1}^{n}\sqrt{\left(\frac{x_1(t_i)-x_1(t_{i-1})}{t_i-t_{i-1}}\right)^2+\dots+\left(\frac{x_n(t_i)-x_n(t_{i-1})}{t_i-t_{i-1}}\right)^2}\,(t_i-t_{i-1}).$$

When we took the limit as $|t_i-t_{i-1}|\to 0$ for all i $(n\to\infty)$ we got derivatives inside and an integral outside to give the formula.

(5 pts) I d) Sketch the curve $F(t) = (\cos t, \sin t, t)$





(10 pts) e) Find its arclength between $t=0$ and $t=2\pi$

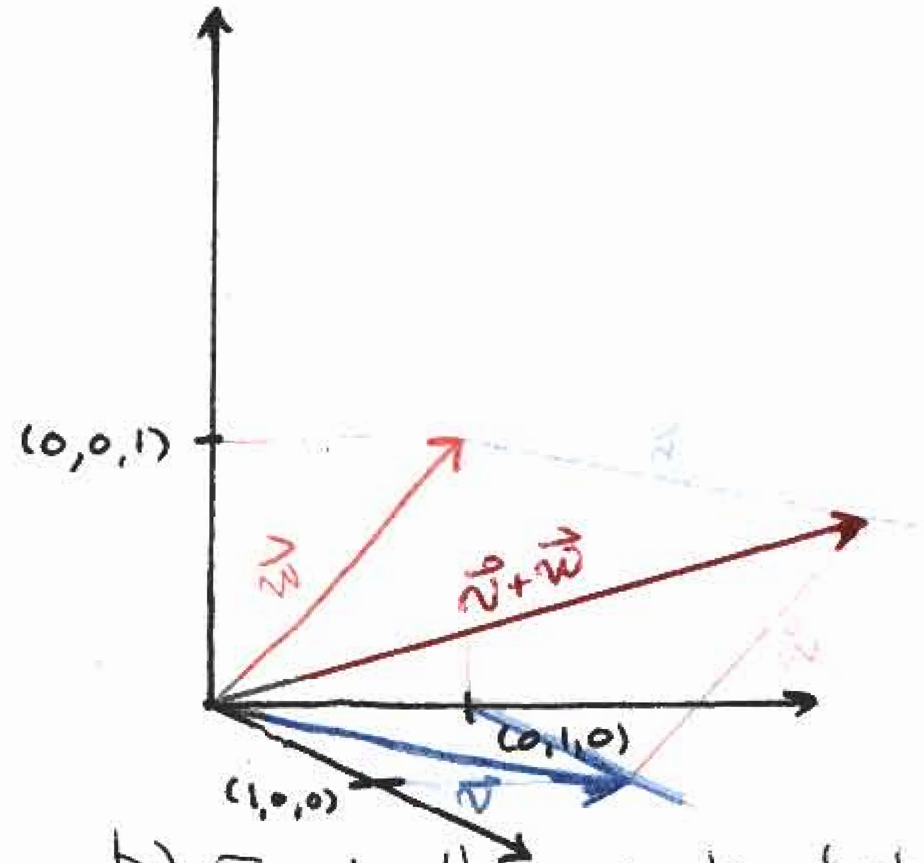
$$F'(t) = (-\sin t, \cos t, 1) = \vec{v}(t)$$

$$v(t) = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1^2} = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$

$$\int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\pi$$

II. a) Draw the vectors $\vec{v}=(1,1,0)$ and $\vec{w}=(0,1,1)$

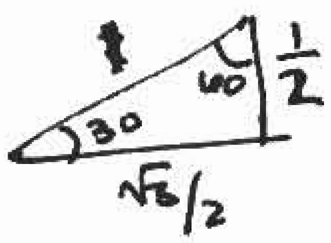
(5 pts) and the vector $\vec{v}+\vec{w}$.



(5 pts) b) Find the angle between $\vec{v}$ and $\vec{w}$

$(1,1,0)\cdot(0,1,1) = \vec{v}\cdot\vec{w} = \|\vec{v}\|\|\vec{w}\|\cos\theta$ $\qquad \|\vec{v}\| = \sqrt{1^2+1^2+0^2} = \sqrt{2} \qquad \|\vec{w}\| = \sqrt{0^2+1^2+1^2} = \sqrt{2}$

$1 = (\sqrt{2})(\sqrt{2})\cos\theta \quad \Rightarrow \quad \cos\theta = \frac{1}{2} \qquad \theta = \arccos\left(\frac{1}{2}\right)$



$\theta = 60^\circ = \frac{\pi}{3}$ radians

(5 pts) c) Find the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.

area of parallelogram $= \|\vec{v}\times\vec{w}\| = \left\|\left(\begin{vmatrix}1&0\\1&1\end{vmatrix}, -\begin{vmatrix}1&0\\0&1\end{vmatrix}, \begin{vmatrix}1&1\\0&1\end{vmatrix}\right)\right\|$

$= \|(1,-1,1)\| = \sqrt{1^2+(-1)^2+1^2} = \sqrt{3}$

(5 pts) III. a) Define linear transformation

A linear transformation is a map $T: \mathbb{R}^n \to \mathbb{R}^m$ such that

$$T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w})$$
$$T(\alpha\vec{v}) = \alpha T(\vec{v})$$

(5 pts) b) The matrix $A = \begin{pmatrix} 1 & 5 & 2 \\ 2 & -1 & 0 \\ 4 & 0 & -2 \end{pmatrix}$ gives a linear transformation T from $\mathbb{R}^3 \to \mathbb{R}^3$.

What is $T(0,1,3)$?

$$\begin{pmatrix} 1 & 5 & 2 \\ 2 & -1 & 0 \\ 4 & 0 & -2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1\cdot 0 + 5\cdot 1 + 2\cdot 3 \\ 2\cdot 0 - 1\cdot 1 + 0\cdot 3 \\ 4\cdot 0 + 0\cdot 1 + 2\cdot 3 \end{pmatrix} = \begin{pmatrix} 11 \\ -1 \\ -6 \end{pmatrix}$$

(5 pts) c) Find the determinant of A.

$$\begin{vmatrix} 1 & 5 & 2 \\ 2 & -1 & 0 \\ 4 & 0 & -2 \end{vmatrix} = 1\begin{vmatrix} -1 & 0 \\ 0 & -2 \end{vmatrix} - 5\begin{vmatrix} 2 & 0 \\ 4 & -2 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix}$$

$$= 1\cdot 2 - 5\cdot(-4) + 2(+4) = 2 + 20 + 8 = 30$$

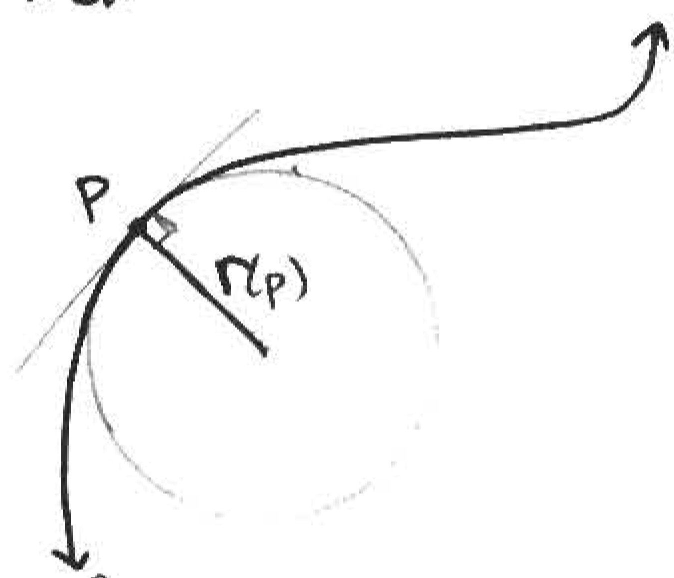
(5 pts) d) What does this say about the transformation T?

Since $\begin{vmatrix} 1 & 5 & 2 \\ 2 & -1 & 0 \\ 4 & 0 & 2 \end{vmatrix}$ = volume of parallelepiped spanned by $T(\vec{i}) = (1,2,4)$
$T(\vec{j}) = (5,-1,0)$
and $T(\vec{k}) = (2,0,2)$,

This says that T multiplies volumes by a factor of 30.

IV a) Define radius of curvature and use a picture to explain its geometric meaning (10 pts)

Def The radius of curvature $r = \frac{1}{\kappa}$, where κ is the curvature. It is the radius of the circle which best approximates the curve at a point:



(5 pts) b) Find the velocity vector for an object whose position is given by $F(t) = (t^2, 1/t)$

$\vec{v}(t) = F'(t) = (2t, -1/t^2)$

(5 pts) c) What happens to the speed of the object as $t \to \infty$?

Speed $= v(t) = \|\vec{v}(t)\| = \|(2t, -1/t^2)\| = \sqrt{4t^2 + 1/t^4} \longrightarrow \infty$

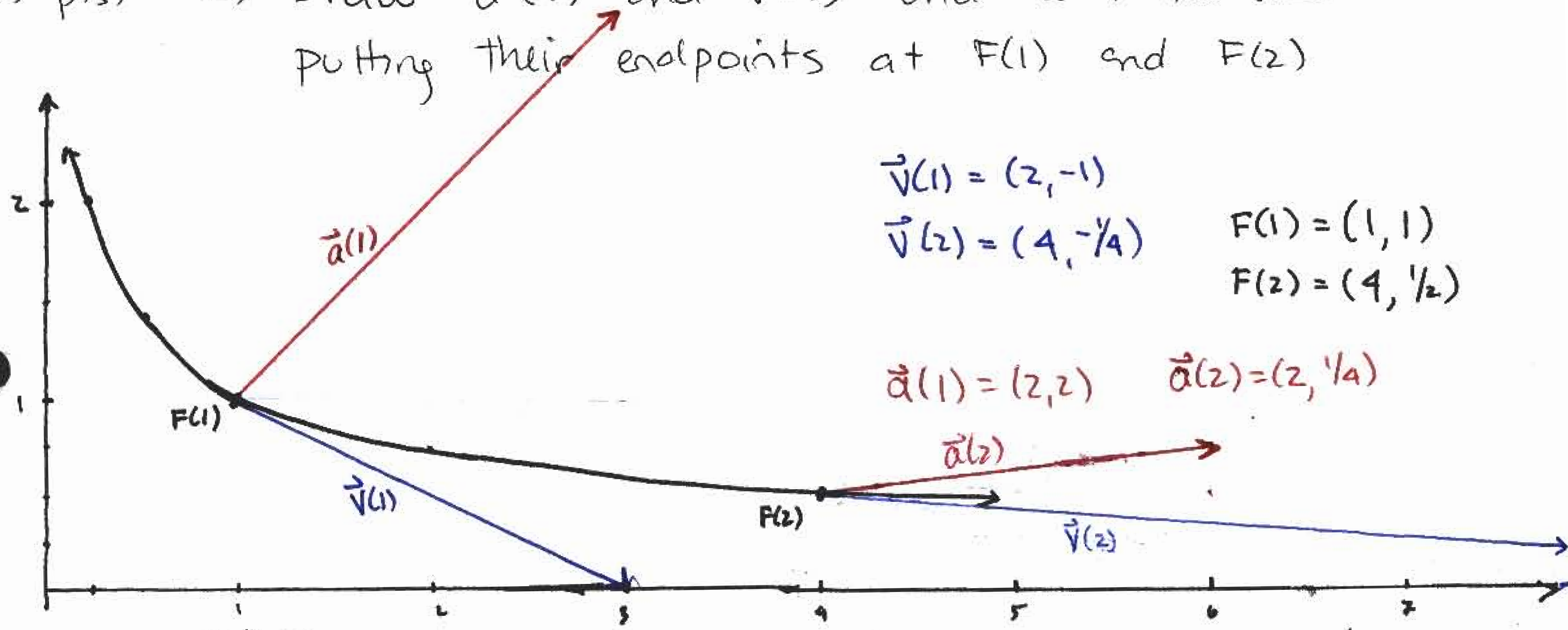
as $t \longrightarrow \infty$

(or $\to \infty$ or $\to 0$)

(5 pts) d) Find the acceleration vector for this object

$\vec{a}(t) = v'(t) = (2, 2/t^3)$

(5 pts) e) Draw $\vec{a}(1)$ and $\vec{v}(1)$ and $\vec{a}(2)$ and $\vec{v}(2)$ putting their endpoints at F(1) and F(2)



(5 pts) f) Find the curvature and radius of curvature for the path along which the object is travelling. Evaluate at t=1 and t=2.

$$K = \frac{\left|\begin{vmatrix} v_1 & v_2 \\ a_1 & a_2 \end{vmatrix}\right|}{v^3} = \frac{\left|\begin{vmatrix} 2t & -1/t^2 \\ 2 & 2/t^3 \end{vmatrix}\right|}{\left(\sqrt{4t^2 + 1/t^4}\right)^3} = \frac{\left|\frac{4}{t^2} + \frac{2}{t^2}\right|}{\sqrt{4t^2 + 1/t^4}^{\,3}} = \frac{6}{t^2\sqrt{4t^2 + 1/t^4}^{\,3}}$$

$$r = \frac{1}{K} = \frac{t^2\sqrt{4t^2 + 1/t^4}^{\,3}}{6}$$

$$K(1) = \frac{6}{1\sqrt{5}^{\,3}} = \frac{6}{5\sqrt{5}} \qquad r(1) = \frac{5\sqrt{5}}{6} \sim 1.86$$

$$K(2) = \frac{6}{4\sqrt{16 + 1/16}^{\,3}} = \frac{3}{2\left(\sqrt{16 + 1/16}\right)^3} \qquad r(2) = \frac{2\sqrt{16 + 1/16}^{\,3}}{3} \sim 342 \qquad \frac{2}{3} \cdot \sqrt{16}^{\,3}$$