MIDTERM EXAM SOLUTIONS

1) The vector space $\mathbb{R}^n$ is the set of ordered n-tuples of real numbers $\mathbb{R}^n = \{\langle x_1, .., x_n \rangle \mid x_i \in \mathbb{R}\}$ together with the operations of vector addition:

$$+ : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$\langle x_1, .., x_n \rangle + \langle y_1, .., y_n \rangle = \langle x_1 + y_1, .., x_n + y_n \rangle$$

and scalar multiplication

$$\cdot : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$c \langle x_1, .., x_n \rangle = \langle c x_1, ..., c x_n \rangle .$$

2) A vector valued function $\vec{r} : I \longrightarrow \mathbb{R}^n$, $I \subseteq \mathbb{R}$, is called <u>smooth</u> if $\vec{r}(t)$ is continuous on I and differentiable with $\vec{r}\,'(t) \neq 0$ for $t \in \text{Interior}(I)$. A curve in $\mathbb{R}^n$ is called <u>smooth</u> if it admits a smooth parametrization.

3) <u>Thm</u> If $\vec{r}(t)$ describes the motion of an object in space, $v(t)$ is its velocity at time t, $K(t)$ is the curvature of its path at time t, $T(t)$ is the unit tangent vector to its path at time t and $N(t)$ is the unit normal vector to its path at time $N(t)$, then the acceleration of the particle decomposes as

$$\vec{a}(t) = v'(t)\vec{T}(t) + K(t)v^2(t)\vec{N}(t).$$

4) <u>Theorem</u> $\frac{d}{dt}[\vec{v}(t)\cdot\vec{w}(t)] = \vec{v}\,'(t)\cdot\vec{w}(t) + \vec{v}(t)\cdot\vec{w}\,'(t)$

<u>proof</u> Let $\vec{v}(t) = \langle v_1(t), \ldots, v_n(t)\rangle$

$\vec{w}(t) = \langle w_1(t), \ldots, w_n(t)\rangle$.

Then $\frac{d}{dt}[\vec{v}(t)\cdot\vec{w}(t)] = \frac{d}{dt}[\langle v_1(t), \ldots, v_n(t)\rangle \cdot \langle w_1(t), \ldots, w_n(t)\rangle]$

$= \frac{d}{dt}[v_1(t)w_1(t) + \cdots + v_n(t)w_n(t)]$ by definition of dot product + sum rule

$= \frac{d}{dt}[v_1(t)w_1(t)] + \cdots + \frac{d}{dt}[v_n(t)w_n(t)]$

$= v_1'(t)w_1(t) + v_1(t)w_1'(t) + \cdots + v_n'(t)w_n(t) + v_n(t)w_n'(t)$

by product rule

$= (v_1'(t)w_1(t) + \cdots + v_n'(t)w_n(t)) + (v_1(t)w_1'(t) + \cdots + v_n(t)w_n'(t))$

by commutativity

$= \langle v_1'(t), \ldots, v_n'(t)\rangle \cdot \langle w_1(t), \ldots, w_n(t)\rangle$

$+ \langle v_1(t), \ldots, v_n(t)\rangle \cdot \langle w_1'(t), \ldots, w_n'(t)\rangle$

by def dot prod.

$= \frac{d}{dt}\langle v_1(t), \ldots, v_n(t)\rangle \cdot \langle w_1(t), \ldots, w_n(t)\rangle$

$+ \langle v_1(t), \ldots, v_n(t)\rangle \cdot \frac{d}{dt}\langle w_1(t), \ldots, w_n(t)\rangle$

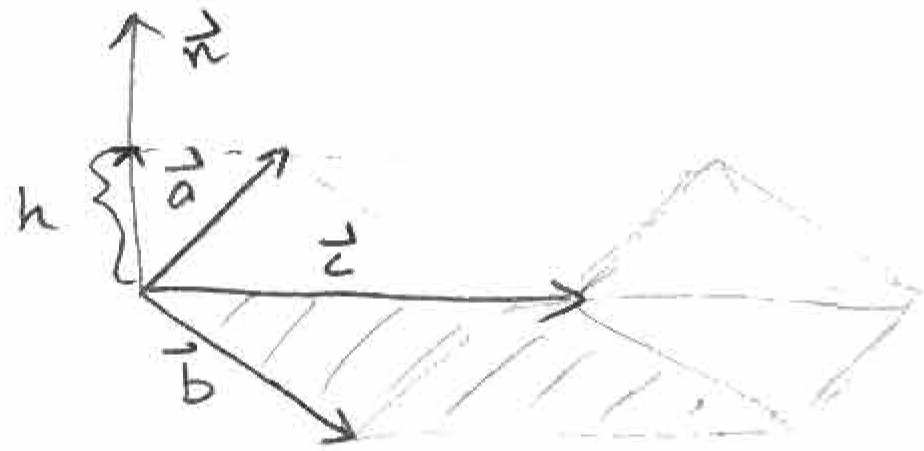
by theorem that derivatives of vector valued functions are taken component-wise

$= \vec{v}\,'(t)\cdot\vec{w}(t) + \vec{v}(t)\cdot\vec{w}\,'(t)$ as required.

$\blacksquare$

5) Theorem $\left\| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} \right\|$ = volume of parallelopiped spanned by $\langle a_1, a_2, a_3 \rangle$, $\langle b_1, b_2, b_3 \rangle$, and $\langle c_1, c_2, c_3 \rangle$.

Proof The volume of the parallelopiped



is Volume = area of base × height

$= \|\vec{b} \times \vec{c}\| \times$ height   by theorem that $\|\vec{b}\times\vec{c}\|$ = area of parallelogram

$= \|\vec{b} \times \vec{c}\| \times \|proj_{\vec{n}} \vec{a}\|$ where $\vec{n} \perp$ the plane of $\vec{b}$ and $\vec{c}$

$= \|\vec{b} \times \vec{c}\| \times \|proj_{\vec{b}\times\vec{c}} \vec{a}\|$ since $\vec{b}\times\vec{c} \perp \vec{b}$ and $\vec{b}\times\vec{c} \perp \vec{c}$

$= \|\vec{b} \times \vec{c}\| \times \left\| \frac{\vec{a}\cdot(\vec{b}\times\vec{c})}{\|b\times\vec{c}\|^2} \vec{b}\times\vec{c} \right\|$ by proj. formula

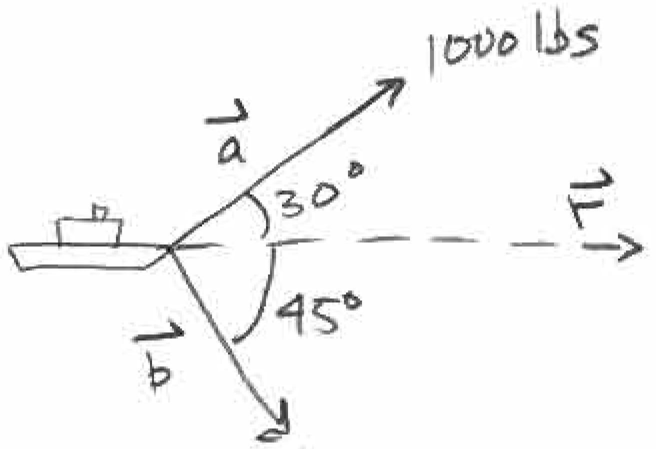
$= \|\vec{b} \times \vec{c}\| \times \frac{|\vec{a}\cdot(\vec{b}\times\vec{c})|}{\|\vec{b}\times\vec{c}\|^2} \|\vec{b}\times\vec{c}\|$ since $\|c\vec{v}\| = |c|\|\vec{v}\|$

$= |\vec{a}\cdot(\vec{b}\times\vec{c})|$

$= \left| \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \right|$ since the triple product is the determinant.

6) Essay: See me if you have questions about how to improve your essay.

7)



We want $\vec{a}+\vec{b}=\vec{r}$. Taking components, we get $\langle 1000\cos 30°, 1000\sin 30°\rangle + \langle \|b\|\cos(-45°), \|b\|\sin(-45°)\rangle = \langle \|r\|, 0\rangle$

So equating components, we get

$$\begin{cases} 1000\cos 30° + \|b\|\cos(-45°) = \|r\| \\ 1000\sin 30° + \|b\|\sin(-45°) = 0 \end{cases}$$

rearrange!

$$\begin{cases} \|r\| - \|b\|\cos(-45°) = 1000\cos 30° \\ \quad\quad - \|b\|\sin(-45°) = 1000\sin 30° \end{cases}$$

This is a linear system of 2 equations in the variables $\|r\|$ and $\|b\|$.

$$\begin{cases} \|r\| - \frac{\|b\|}{\sqrt{2}} = 500\sqrt{3} \\ \quad\quad \frac{\|b\|}{\sqrt{2}} = 500 \end{cases} \Rightarrow \begin{cases} \|r\| = 500\sqrt{3} + \frac{\|b\|}{\sqrt{2}} \\ \|b\| = 500\sqrt{2} \end{cases}$$

So $\|b\| = \overset{707}{500\sqrt{2}}$ lbs is the force required by boat B and $\|r\| = 500\sqrt{3} + 500$ is the resultent force.

8) $\vec{r}(t) = \langle r_x(0) + v_x(0)\cdot t,\ r_y(0) + v_y(0)t - \frac{a}{2}t^2 \rangle$

where $\vec{r}(0) = \langle r_x(0), r_y(0) \rangle$ is the initial position
$\vec{v}(0) = \langle v_x(0), v_y(0) \rangle$ is the initial velocity
and a = acceleration due to gravity.

Here, $\vec{r}(0) = \langle 0, 6 \rangle$
$\vec{v}(0) = \langle 50\cos 45°, 50 \sin 45° \rangle = \langle 25\sqrt{2}, 25\sqrt{2} \rangle$
$a = 12$

so $\vec{r}(t) = \langle 25\sqrt{2}\, t,\ 6 + 25\sqrt{2}\, t - 6t^2 \rangle$

To find how far it goes, determine when it lands: t_L is time where $r_y(t_L) = 0$:

$$6 + 25\sqrt{2}\, t_L - 6t_L^2 = 0$$

$$t_L^2 - \frac{25\sqrt{2}}{6} t_L - 1 = 0$$

$$t_L = \frac{\frac{25\sqrt{2}}{6} + \sqrt{\left(\frac{25\sqrt{2}}{6}\right)^2 + 4}}{2} = \frac{25\sqrt{2} + \sqrt{(25\sqrt{2})^2 + 4\cdot 36}}{12} \approx$$ seconds

Now see how far it went by t_L:
$r_x(t_L) = r_x(6.06) = 25\sqrt{2}\cdot 6.06 \approx 214$ feet

9) a) $f'(t) = \langle -\sin t, \cos t, 3 \rangle$

At the point $\langle \sqrt{3}/2, 3/2, \pi/2 \rangle = \langle \cos t, \sin t + 1, 3t \rangle$, $t = \pi/6$, so $f'(\pi/6) = \langle -\sin \pi/6, \cos \pi/6, 3 \rangle$

$$= \langle -\tfrac{1}{2}, \sqrt{3}/2, 3 \rangle$$

so the parametric equation of the tangent line is:

$$\vec{r}(t) = \langle \sqrt{3}/2, 3/2, \pi/2 \rangle + t \langle -1/2, \sqrt{3}/2, 3 \rangle$$
$$= \langle \sqrt{3}/2 - \tfrac{1}{2}t, 3/2 + \sqrt{3}/2\, t, \pi/2 + 3t \rangle$$

b) The curvature at $t = \pi/6$ is

$$K(\tfrac{\pi}{6}) = \frac{\| f'(\pi/6) \times f''(\pi/6) \|}{\| f'(\pi/6) \|^3}$$

$$f''(t) = \langle -\cos t, -\sin t, 0 \rangle$$
$$f''(\pi/6) = \langle -\sqrt{3}/2, -1/2, 0 \rangle$$

So $$f'(\pi/6) \times f''(\pi/6) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1/2 & \sqrt{3}/2 & 3 \\ -\sqrt{3}/2 & -1/2 & 0 \end{vmatrix}$$

$$= \langle 3/2, -3\sqrt{3}/2, \tfrac{1}{4} + \tfrac{3}{4} \rangle$$
$$= \langle 3/2, -3\sqrt{3}/2, 1 \rangle$$

so $$\frac{\| f'(\pi/6) \times f''(\pi/6) \|}{\| f'(\pi/6) \|^3} = \frac{\sqrt{9/4 + \frac{27}{4} + 1}}{\sqrt{\frac{1}{4} + \frac{3}{4} + 9}^{\,3}} = \frac{\sqrt{10}}{(\sqrt{10})^3} = \frac{1}{10}$$

c) The Length of the curve between
$f(0) = \langle 1, 1, 0 \rangle$ and $f(\pi/6) = (\sqrt{3}/2, 3/2, \pi/2)$

is $\displaystyle\int_0^{\pi/6} \|f'(t)\|\, dt = \int_0^{\pi/6} \|\langle -\sin t, \cos t, 3 \rangle\|\, dt$

$$= \int_0^{\pi/6} \sqrt{\sin^2 t + \cos^2 t + 3^2}\, dt = \int_0^{\pi/6} \sqrt{10}\, dt$$

$$= \sqrt{10}\, t \Big|_0^{\pi/6} = \frac{\sqrt{10}\,\pi}{6}$$

10) The point of intersection of the planes

$$\begin{cases} x + 2y + 3z = 2 \\ 2x + 5y + 3z = -1 \\ x \qquad\;\; + 8z = 0 \end{cases}$$

is the point (x, y, z) that simulteneously solves these equations.

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 2 & 5 & 3 & -1 \\ 1 & 0 & 8 & 0 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -3 & -5 \\ 1 & 0 & 8 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -3 & -5 \\ 0 & -2 & 5 & -2 \end{array}\right) \xrightarrow{R_3 \to R_3 + 2R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & -1 & -12 \end{array}\right)$$

$$\xrightarrow{R_3 \to -R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 1 & 12 \end{array}\right) \xrightarrow{R_2 \to R_2 + 3R_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 0 & 31 \\ 0 & 0 & 1 & 12 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - 3R_3} \left(\begin{array}{ccc|c} 1 & 2 & 0 & -34 \\ 0 & 1 & 0 & 31 \\ 0 & 0 & 1 & 12 \end{array}\right) \xrightarrow{R_1 \to R_1 - 2R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -96 \\ 0 & 1 & 0 & 31 \\ 0 & 0 & 1 & 12 \end{array}\right)$$

so the point of intersection is $(-96, 31, 12)$.