FINAL EXAM A 105

CALC 16

HonorPledge ______ NAME_ ______

No aids such as notes, books or calculators allowed on this exam. There are 110 points possible. I will grade out of 100.

I. (10 points) Explain why, if the position of an object in the plane is given by $F(t)$, its velocity is given by $F'(t)$.

In order to justify that $F'(t)$ is the velocity, I must show that $F'(t)$ has the right magnitude and the right direction to be the velocity (Since velocity is a vector of both direction & magnitude) 10/10

We know that if we have a curve C in $\mathbb{R}^n$ that is parametrized $F(t) = (x_1(t) \ldots x_n(t))$, then if f is differentiable the distance from $F(a)$ to $F(b)$ is the arclength given $\int_a^b \sqrt{\frac{dx_1(t)}{dt}^2 \ldots \frac{dx_n(t)}{dt}^2}$. Then we know the speed $= \frac{d}{dt} \int_a^b \sqrt{\frac{dx_1(t)}{dt}^2 \ldots \frac{dx_n(t)}{dt}^2}$. By FTC I, this $= \sqrt{\frac{x_1'(t)}{dt}^2 \ldots \frac{x_n'(t)}{dt}^2}$ which is

$\left\| \frac{dx_1(t)}{dt} \ldots \frac{dx_n(t)}{dt} \right\|$ which is the magnitude of the velocity vector or the speed.

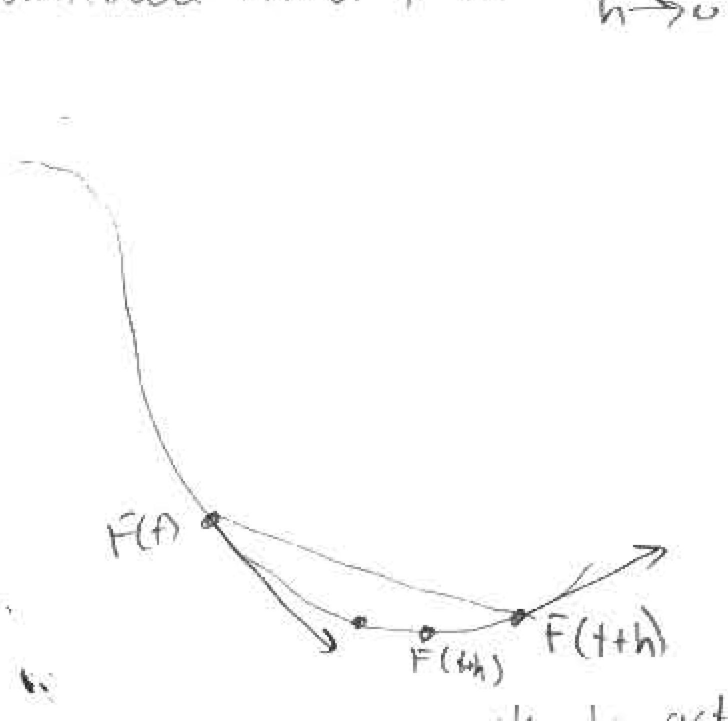
Now I've shown $F'(t)$ has the right magnitude, but what about direction?

We know the derivative of $F(t) = (x_1(t) \ldots x_n(t)) = F'(t) = x_1'(t) \ldots x_n'(t)$ (can distribute limit) or $\lim_{h \to 0} \frac{F(t+h) - F(t)}{h}$

Note

if $h > 0$, the direction stays the same.

$h < 0$, would simply change direction



make h get smaller

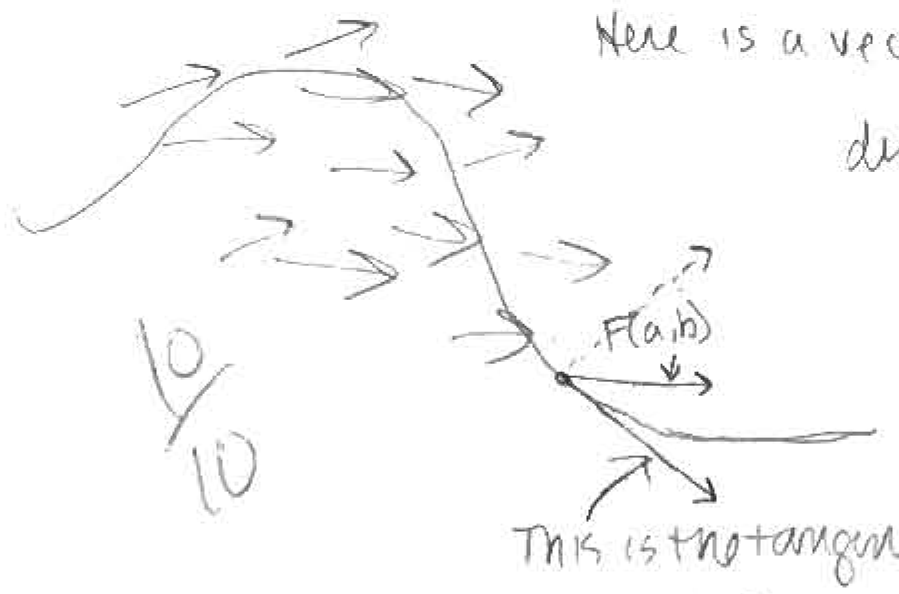
By taking h small, we get a better + better approx. and it is going to the zero vector.

Therefore $F'(t)$ is the right direction.

So Velocity is given by $F'(t)$.

II a) (10 points) Explain how to find the work done by a force $\vec{F}(x,y)$ on a particle moving along a path, $\mathcal{C}$. Why is this the correct formula?

Work is defined as force times distance but this definition is vague. Force means the magnitude of force in the direction of movement.



10/10

Here is a vector field of forces. Since only the force in the direction of movement counts, the force in the tangential direction is what we care about.

The projection of $\vec{F}(a,b)$ onto $\vec{T}$ is the magnitude we want.

$$\|\text{proj}_{\vec{T}}\vec{F}(a,b)\| = \left|\frac{\|\vec{T}(a,b)\|}{\|\vec{T}\|} \cdot \vec{F}(a,b)\right|$$

In order to keep track of the signs we take off the absolute values + write it

$$\frac{\vec{T}(a,b)}{\vec{T}} \cdot \vec{F}(a,b) = \text{projection}.$$

Now we break up the path + add up all the small sections by Riemann sums.

$$\sum_{i=1}^{n} \frac{\vec{T}(a,b)}{\|\vec{T}\|} \cdot \vec{F}(a,b)$$

To better this approximation, take the integral

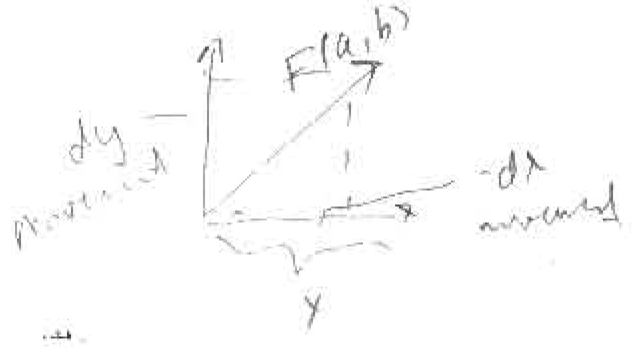
$$\int_C \frac{T(x(t),y(t))}{\sqrt{x'(t)^2+y'(t)^2}} \cdot \vec{F}(x(t),y(t))\, ds = \int_C \frac{T(x(t),y(t))}{\cancel{\sqrt{x'(t)^2+y'(t)^2}}} \cdot \vec{F}(x(t),y(t)) \cdot \underbrace{\cancel{\sqrt{x'(t)^2+y'(t)^2}}}_{ds}$$ and

The tangential component is $x'(t), y'(t)$ so $\int_C (x'(t), y'(t)) \cdot \vec{T}(x(t), y(t))$

so $\int_C f_1 x'(t) + f_2(y'(t)) = \int_C f_1\, dx + \int_C f_2\, dy.$

In this case dx is the x movement and dy is the y movement.

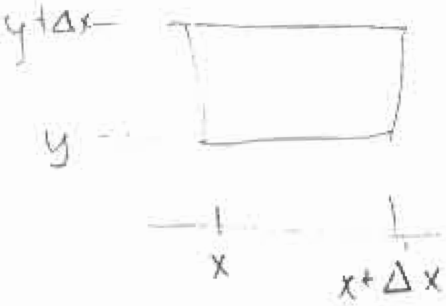
We can write $W = \int \vec{F} \cdot dr$ if $dr = dx\, dy$.



(10 pts)

III. Explain why, in polar coordinates, $dA = r\,dr\,d\theta$

First, we can compare polar coordinates to rectangular coordinates.



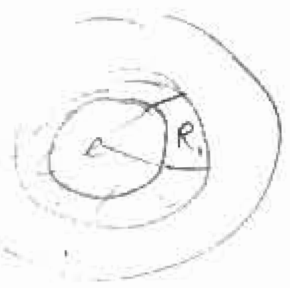
In this case area $= \Delta x \Delta y$. so $dA = dx\,dy$.

10/10

We know that radians is defined as $\frac{\theta}{2\pi}$ so a r segment of a circle is length $\frac{\theta}{2\pi} \cdot$ circumference $= \frac{\theta}{\cancel{2\pi}} \cdot \cancel{2\pi} r$ so $= r\theta$ therefore, we can make a heuristic argument that Area $= r\,dr\,d\theta$. It wouldn't just be $dr\,d\theta$ because since $d\theta$ is in radians, units come out to be just in length. We need another r to get units in area.

Better Argument:

We know the area of the annulus = area of big disk − area of little disk



so we can write

Area

$$\text{Area of } R_1 = \frac{\Delta\theta}{2\pi} \cdot \left(\pi (r+\Delta r)^2 - \pi r^2\right)$$

so simplifying $\frac{\Delta\theta}{2}\left((r+\Delta r)^2 - r^2\right)$

$$dA = \cancel{r^2} + 2\Delta r \cdot r + \Delta r^2 - \cancel{r^2} \qquad \frac{\Delta\theta}{2}\left(2\Delta r \cdot r + \Delta r^2\right)$$

$= \Delta\theta \Delta r\, r \left(1 - \frac{\Delta r}{2\Delta\theta}\right)$ and as $\Delta r \to 0$, $1 - \frac{\Delta r}{2\Delta\theta}$ goes to 1 so

$$\Delta\theta \cdot \Delta r \cdot r\,(1) = r\,\Delta r\,\Delta\theta = dA$$

(15 points)

IV. State and justify the lagrange multiplier method for finding extrema of a function of two variables and subject to one constraint.

To find the extrema of a function $f(x,y)$ subject to the constraint $g(x,y)=0$, if f has continuous partial derivatives and g is differentiable, then ~~there is a~~ points (x,y,λ) such that $\nabla h(x,y,\lambda)=0$ ~~and~~ $h(x,y,\lambda)=f(x,y)-\lambda(g(x,y))$ where ok

find

We have a parametrization of $g(x,y)=0$, where $F(t)=(x(t),y(t))$.

We can plug this parametrization into $f(x,y)$ to get ~~it~~ in one variable of u so $u=(f(x(t),y(t)))$. Now we want $\frac{du}{dt}=0$ so

$$\frac{du}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}=\nabla f\cdot F'(t).$$

So since we want $\frac{du}{dt}=0$ the $\nabla f\perp F'(t)$.

Since we know that it is paramelized by arc length, any point on the curve $g(x,y)=0$ will be $\nabla g\perp F'(t)$ so then we know $\nabla g \parallel \nabla f$. If there are parallel, they must differ only by a constant λ. These are all true if both

$$\nabla f=\lambda\nabla g$$
$$g(x,y)=0$$

are satisfied.

15/15

So we can put these together to form $h(x,y,\lambda)=f(x,y)-\lambda(g(x,y))$

To solve:

$\frac{\partial h}{\partial x}=0$ then Solve for the unknowns

$\frac{\partial h}{\partial y}=0$

$\frac{\partial h}{\partial \lambda}=0=-g(x,y)$ ← (negative constraint)

(10 points)

V. a) Define curvature of a curve in $\mathbb{R}^2$. Use pictures to expl

Let $F(t)$ be in $\mathbb{R}^n$ + parametrized by arclength. Let $k(\theta)$ be the angle between $F'(t)$ and $(1,0)$, then the curvature $k = \left| \frac{dk(\theta)}{dt} \right|$.

9/10

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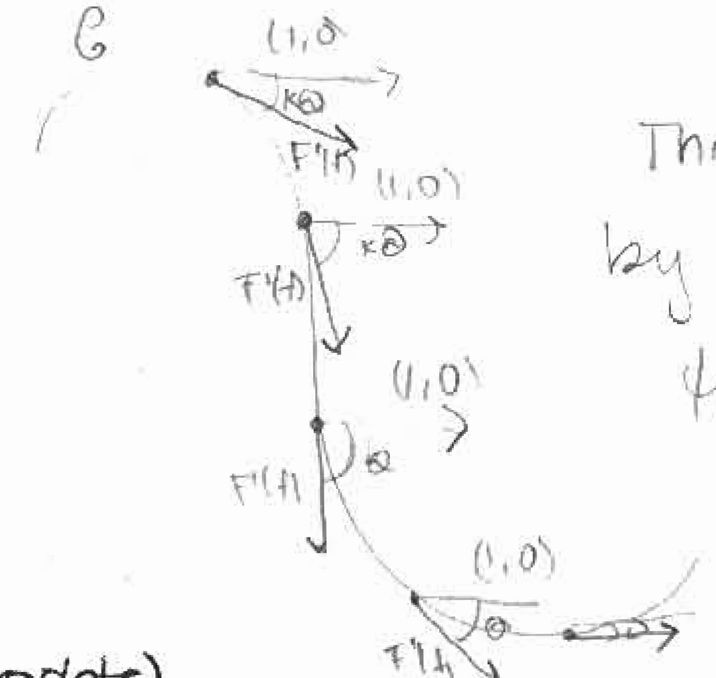
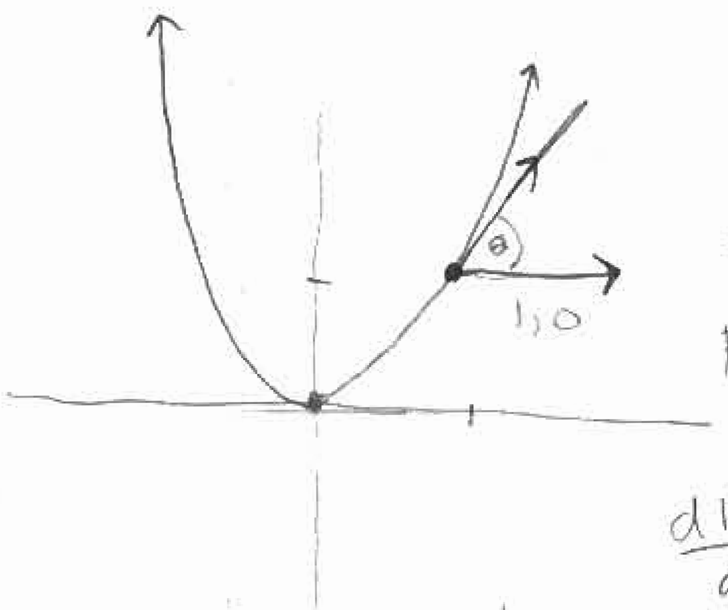


This shows how quickly the curvature is changing by comparing the tangent vectors along the curve to the unit vector $(1,0)$

(5 points)

b) Find the curvature at $(1,1)$ of the parabola $y=x^2$.



* y is held constant

$F(x,y) = x^2$

Need to parameterize: $F(x(t),y(t)) = (t^2, 0)$

$F(x(t),y(t)) = t^2, 0$

$F(x'(t),y'(t)) = 2t\,dt$

Not arclength param

So slope of tangent line @ $(1,1) = 2$

$\frac{dk(\theta)}{dt}$

The angle between $\vec{T}$ and $(1,1) = k\theta$ which is $22.5°$ so $\frac{d(22.5°)}{dt}$

or $\frac{3\pi}{8} = \theta$

Curvature = 2 since 2nd derivative = 2

I know curvature deals with the second derivative and if the second derivative is +, it is concave up and if 2nd derivative is negative it is concave down,

$F''(x,y) = 2$ so, we know curvature is concave up.

2/5

(10 points)

VI. Show that any function of the form $g(x,t) = f(x+kt)$ satisfies the wave equation $(f: \mathbb{R} \to \mathbb{R},\ g: \mathbb{R}^2 \to \mathbb{R})$.

$g(x,t) = f(x+kt)$

The wave equation says that $\frac{\partial f^2}{\partial^2 x} = k \frac{\partial f^2}{\partial t}$ which says the curvature is proportional to the acceleration

For any function: (chain rule)

$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$ ✓✓

$g(x,t) = f(x+kt)$

I will use the example to show this is true

$g(x,t) = \sin(x+kt)$

$\frac{\partial f}{\partial x}$ $f'(x+kt)$ $f''(x+kt)$

$\frac{\partial f}{\partial t}$: $kf(x+kt)$ $k^2 f''(f+kt)$

differ by a constant

$\frac{\partial g}{\partial x} = \cos(x+kt)$ $\quad \frac{\partial g^2}{\partial x^2} = -\sin(x+kt)$

$\frac{\partial g}{\partial t} = k\cos(x+kt)$

$\frac{\partial g^2}{\partial t^2} = -k^2\sin(x+kt)$

so $\frac{\partial g^2}{\partial x} = -\sin(x+kt)$

$\frac{\partial g^2}{\partial t^2} = -k^2\sin(x+kt)$

} as you can see, these only differ by a constant!

10/10

~~constant~~

(5 points)

VII. a) Define divergence of a vector field.

Divergence of a differentiable vector field goes to a scalar field so $f: \mathbb{R}^2 \to \mathbb{R}$

$\nabla f = \frac{\partial f_1}{\partial x}(a,b) + \frac{\partial f_2}{\partial y}(a,b) =$ divergence

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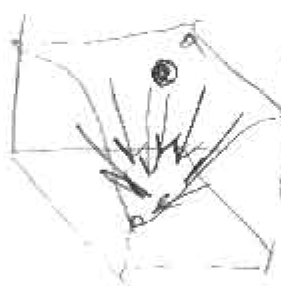


tendancy for ball to roll w/ gravity

(5 points)

b) What is the meaning of divergence of $\vec{F}$ if f is flow of a fluid?

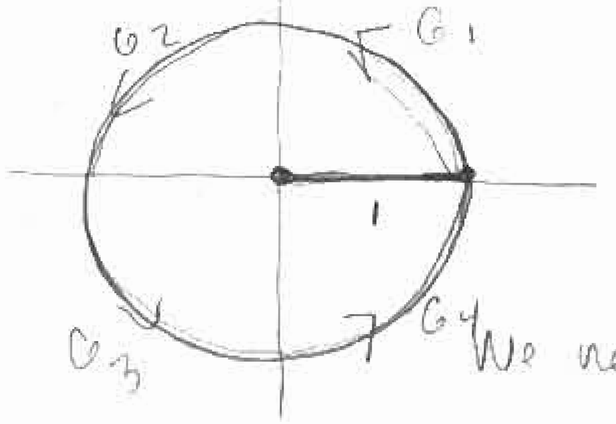
If f is the flow of a fluid, then the divergence of $\vec{F}$ is the tendancy for that fluid to be produced or absorbed at a point

$\int_p (\text{production})\, dA$?

5/5

(10 points)

c) Using a line integral, calculate the flux of the flow $\vec{F}(x,y) = (2y^2, x)$ through the unit circle centered at $(0,0)$. ~~[illegible]~~



$y = x$

$\vec{F}(x,y) = (2y^2, x)$ $x = 2y^2$

$\sqrt{x^2+y^2} = 1$

$F(x(t), y(t))\ (\cos t, \sin t)$

$F(x'(t), y'(t)) = (-\sin t, \cos t)$

We need to parametrize

$\text{Flux} = \int_C P\,dy - \int_C Q\,dx$

$\int_C f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}\, ds$

$\int_C P\,dy - \int_C Q\,dx \qquad \int 2x^2\,dy - \int y\,dx$

$\cancel{\int_0^{\pi/2} f(\cos t, \sin t)\,ds + \int_{\pi/2}^{\pi} f(\cos t, \sin t)\,ds + \int_{\pi}^{3\pi/2} f(\cos t, \sin t)\,ds + \int_{3\pi/2}^{2\pi} f(\cos t, \sin t)\,ds}$

$\frac{ds}{dt} = 1$ for all sin $\sqrt{\cos^2 t + \sin^2 t} = 1$

$\frac{2}{3}x^3\Big]_0^{2\pi} - \frac{y^2}{2}\Big]_0^{2\pi}$ $\quad x = \cos t,\ y = \sin t \quad$ $\frac{2}{3}(\cos t)^3\Big]_0^{2\pi} - \frac{\sin^2 t}{2}\Big]_0^{2\pi}$

$\left(\frac{2}{3} - \frac{2}{3}\right)_0 - 0 = \boxed{0}$?

✓ 10/10

(5 points)

d) What is the divergence of $\vec{F}(x,y) = (2y^2, x)$? Use Green's Theorem to calculate the flux in c).

$\cancel{\int P\,dy - \int Q\,dx}$ $\quad P = 2y^2,\ Q = x$

$\int_A (\text{Production})\,dA = \int_A \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right) dA =$ Greens Theorem

5/5

$\cancel{\int 2y^2\,dy - \int x\,dx}$

$x = 2y^2 \qquad \frac{\partial f}{\partial x} = 0 \qquad \int_A 0\,dA = 0$

$y = x \qquad \frac{\partial f}{\partial y} = 0$

$f = (2y^2, x)$

Divergence $= 0$

$\cancel{\int x\,dy - \int y\,dx}$

$t,$

$(2t^2, t)$

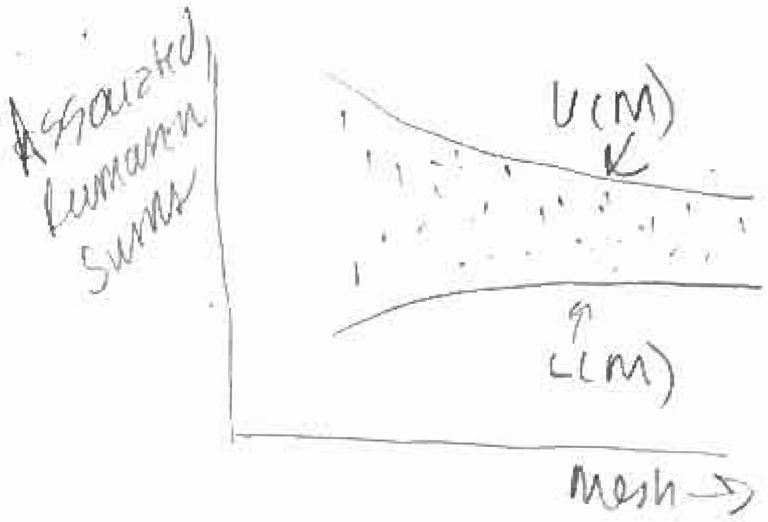
VIII. (10 points) Define the definite integral over $R \subset \mathbb{R}^2$ of $f(x,y)$. Explain this definition using pictures and diagrams.

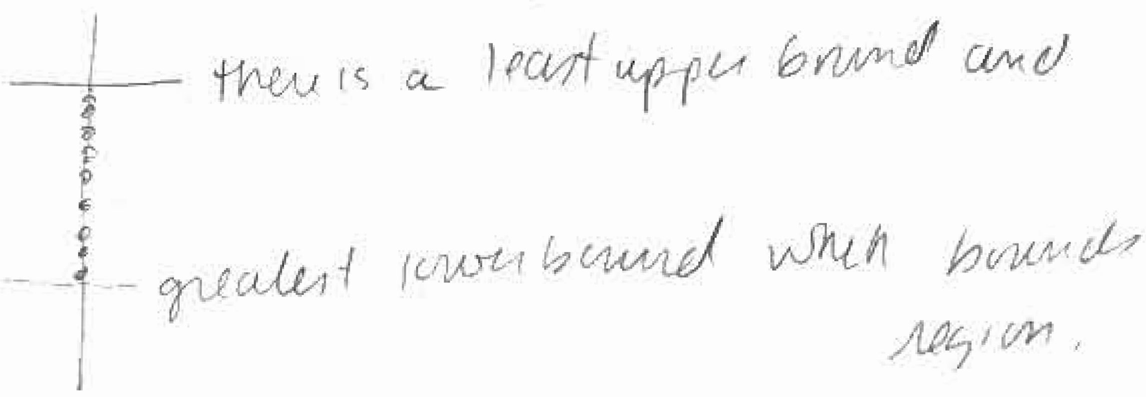
If f is bounded in the bounded region R then define:

$$L(M) = \underset{|P|=M}{\text{greatest lower bound}} \sum_{i=1}^{n} f(\vec{p}_i)\,\text{Area}\,R_i$$

$$U(M) = \underset{|P|=M}{\text{least upper bound}} \sum_{i=1}^{n} f(\vec{p}_i)\,\text{Area}\,R_i$$

If $\lim_{M\to 0} L(M) = k = \lim_{M\to 0} U(M)$, then f is integrable over R and the integral is k which is denoted $\int_R f(\vec{p})\,dA$.





10/10

Derivatives, Integrals, Taylor Series, Limits, function of many variable fns

+5

EXTRA (up to 5 points)
What unifies the concepts we talked about this year in calculus? What <u>were</u> the main concepts? Name at least 3. (I would say there were 6 <u>main</u> concepts)

On Back

This year in the calculus sequence we talked about many fundamental concepts of calculus. We started the year with limits and from those we studied derivatives. By the Fundamental Theorems of calculus, differentiation was linked to a totally different concept of integration. Differentiation and integration are the core topics for they deal with breaking up curves, lines, areas, regions, volumes, etc and making approximations better and better; This is huge in calculus.

We also talked about series and Taylor approximations. These series allowed us to approximate infinitely long functions or polynomials. Again, we went from an estimate or approximation to the exact values of the series.

This third term we learned about Vectors and applying vectors to physical applications. With vectors, derivatives, integrals, and limits we were able to learn the basics behind movement of particles in space and line integrals etc.

Approximation + improving approximations is what unifies these calculus concepts.