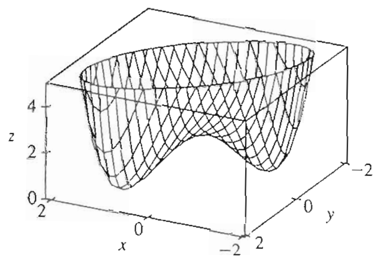
HW 8

14.7

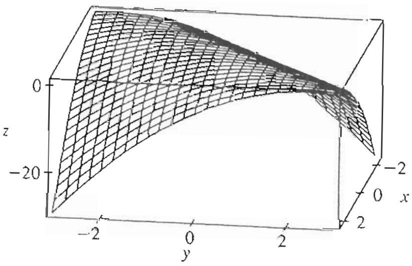
3. In the figure, a point at approximately $(1, 1)$ is enclosed by level curves which are oval in shape and indicate that as we move away from the point in any direction the values of f are increasing. Hence we would expect a local minimum at or near $(1, 1)$. The level curves near $(0, 0)$ resemble hyperbolas, and as we move away from the origin, the values of f increase in some directions and decrease in others, so we would expect to find a saddle point there.

To verify our predictions, we have $f(x, y) = 4 + x^3 + y^3 - 3xy \;\Rightarrow\; f_x(x, y) = 3x^2 - 3y$, $f_y(x, y) = 3y^2 - 3x$. We have critical points where these partial derivatives are equal to 0: $3x^2 - 3y = 0$, $3y^2 - 3x = 0$. Substituting $y = x^2$ from the first equation into the second equation gives $3(x^2)^2 - 3x = 0 \;\Rightarrow\; 3x(x^3 - 1) = 0 \;\Rightarrow\; x = 0$ or $x = 1$. Then we have two critical points, $(0, 0)$ and $(1, 1)$. The second partial derivatives are $f_{xx}(x, y) = 6x$, $f_{xy}(x, y) = -3$, and $f_{yy}(x, y) = 6y$, so
$D(x, y) = f_{xx}(x, y)\, f_{yy}(x, y) - [f_{xy}(x, y)]^2 = (6x)(6y) - (-3)^2 = 36xy - 9$. Then
$D(0, 0) = 36(0)(0) - 9 = -9$, and $D(1, 1) = 36(1)(1) - 9 = 27$. Since $D(0, 0) < 0$, f has a saddle point at $(0, 0)$ by the Second Derivatives Test. Since $D(1, 1) > 0$ and $f_{xx}(1, 1) > 0$, f has a local minimum at $(1, 1)$.

7. $f(x, y) = x^4 + y^4 - 4xy + 2 \;\Rightarrow\; f_x = 4x^3 - 4y$, $f_y = 4y^3 - 4x$, $f_{xx} = 12x^2$, $f_{xy} = -4$, $f_{yy} = 12y^2$. Then $f_x = 0$ implies $y = x^3$, and substitution into $f_y = 0 \;\Rightarrow\; x = y^3$ gives $x^9 - x = 0 \;\Rightarrow\; x(x^8 - 1) = 0 \;\Rightarrow\; x = 0$ or $x = \pm 1$. Thus the critical points are $(0, 0)$, $(1, 1)$, and $(-1, -1)$. Now $D(0, 0) = 0 \cdot 0 - (-4)^2 = -16 < 0$, so $(0, 0)$ is a saddle point. $D(1, 1) = (12)(12) - (-4)^2 > 0$ and $f_{xx}(1, 1) = 12 > 0$, so $f(1, 1) = 0$ is a local minimum. $D(-1, -1) = (12)(12) - (-4)^2 > 0$ and $f_{xx} = (-1, -1) = 12 > 0$, so $f(-1, -1) = 0$ is also a local minimum.



11. $f(x, y) = 1 + 2xy - x^2 - y^2 \;\Rightarrow\; f_x = 2y - 2x$, $f_y = 2x - 2y$, $f_{xx} = f_{yy} = -2$, $f_{xy} = 2$. Then $f_x = 0$ and $f_y = 0$ implies $x = y$ so the critical points are all points of the form (x_0, x_0). But $D(x_0, x_0) = 4 - 4 = 0$ so the Second Derivatives Test gives no information. However $1 + 2xy - x^2 - y^2 = 1 - (x - y)^2$ and $1 - (x - y)^2 \leq 1$ for all (x, y), with equality if and only if $x = y$. Thus $f(x_0, x_0) = 1$ are local maxima.



29. $f_x(x, y) = 2x + 2xy$, $f_y(x, y) = 2y + x^2$, and setting $f_x = f_y = 0$ gives $(0, 0)$ as the only critical point in D, with $f(0, 0) = 4$.
On L_1: $y = -1$, $f(x, -1) = 5$, a constant.
On L_2: $x = 1$, $f(1, y) = y^2 + y + 5$, a quadratic in y which attains its maximum at $(1, 1)$, $f(1, 1) = 7$ and its minimum at $\left(1, -\frac{1}{2}\right)$, $f\left(1, -\frac{1}{2}\right) = \frac{17}{4}$.
On L_3: $f(x, 1) = 2x^2 + 5$ which attains its maximum at $(-1, 1)$ and $(1, 1)$ with $f(\pm 1, 1) = 7$ and its minimum at $(0, 1)$, $f(0, 1) = 5$.
On L_4: $f(-1, y) = y^2 + y + 5$ with maximum at $(-1, 1)$, $f(-1, 1) = 7$ and minimum at $\left(-1, -\frac{1}{2}\right)$, $f\left(-1, -\frac{1}{2}\right) = \frac{17}{4}$. Thus the absolute maximum is attained at both $(\pm 1, 1)$ with $f(\pm 1, 1) = 7$ and the absolute minimum on D is attained at $(0, 0)$ with $f(0, 0) = 4$.