Calc I, HW #5
Selected solutions

3.5: 75

Use the Chain Rule to prove the following:

a) The derivative of an odd function is an even func

b) The derivative of an even function is an odd functi

a) If f is even then

$f(x) = f(-x)$

So differentiate both sides:

$\frac{d}{dx}[f(x)] = \frac{d}{dx}[f(-x)]$

let $u = -x$

$f'(x) = \frac{d}{dx}[f(u)]$

and $y = f(u)$

$f'(x) = \frac{dy}{dx}$

then by the chain rule

$f'(x) = \frac{dy}{du}\frac{dy}{dx}$

$f'(x) = f'(u)(-1)$

$f'(x) = -f'(-x)$

But this says f' is odd.

b) If f is odd, then

$f(x) = -f'(-x)$

$\frac{d}{dx}[f(x)] = \frac{d}{dx}[-f'(-x)]$

$f'(x) = -\frac{d}{dx}[f'(-x)]$

$f'(x) = -(-f'(-x))$ as in (a

$f'(x) = f'(-x)$

So f' is even.

3.6: 40 Prove using implicit differentiation on $y^q = x^p$ that $\frac{d}{dx}\left[x^{\frac{p}{q}}\right] = \frac{p}{q} x^{\frac{p}{q}-1}$.

pf We know $y = f(x) = x^{\frac{p}{q}}$ if $y^q = x^p$.
We want to show $\frac{dy}{dx} = \frac{p}{q} x^{\frac{p}{q}-1}$.
So $\frac{d}{dx}[y^q] = \frac{d}{dx}[x^p]$
Let $u = y^q$. Then we have
$$\frac{du}{dx} = px^{p-1}$$
By the chain rule,
$$\frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = px^{p-1}$$
$$qy^{q-1}\frac{dy}{dx} = px^{p-1}$$

But $y = x^{p/q}$ so sub in:

$$q(x^{p/q})^{q-1}\frac{dy}{dx} = px^{p-1}$$

now solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{p}{q}\frac{x^{p-1}}{(x^{p/q})^{q-1}} = \frac{p}{q}\frac{x^{p-1}}{\left(x^{\frac{p(q-1)}{q}}\right)}$$ by exponent property $(x^a)^b = x^{ab}$

$$= \frac{p}{q}\left[x^{(p-1)-\frac{p}{q}(q-1)}\right]$$ by exponent property $\frac{x^a}{x^b} = x^{a-b}$

$$= \frac{p}{q}\left[x^{\frac{(p-1)q}{q}-\frac{p(q-1)}{q}}\right]$$ adding fractions

$$= \frac{p}{q}\left[x^{\frac{(pq-q)-(pq-p)}{q}}\right]$$ distributing

$$= \frac{p}{q}\left[x^{\frac{-q+p}{q}}\right] = \frac{p}{q}x^{\frac{p}{q}-1}$$ ✓ simplifying

3.7 50

A particle moves along x axis and

$x(t) = \frac{t}{1+t^2}$ $t \geq 0$, t = seconds, x = meters.

a) find acceleration at time t. When is it 0?

$$\text{velocity} = \frac{dx}{dt} = \frac{d}{dt}\left[\frac{t}{1+t^2}\right] = \frac{\frac{d}{dt}[t](1+t^2) - t\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$= \frac{1(1+t^2) - t(2t)}{1+2t^2+t^4} = \frac{-t^2+1}{t^4+2t^2+1}$$

$$\text{acc} = \frac{d\,\text{velocity}}{dt} = \frac{d^2x}{dt} = \frac{d}{dt}\left[\frac{-t^2+1}{t^4+2t^2+1}\right]$$

$$= \frac{\frac{d}{dt}[-t^2+1](t^4+2t^2+1) - (-t^2+1)\frac{d}{dt}(t^4+2t^2+1)}{(t^4+2t^2+1)^2}$$

$$= \frac{(-2t)(t^4+2t^2+1) - (1-t^2)(4t^3+4t)}{((1+t^2)^2)^2}$$

$$= \frac{-2t^5 - \cancel{4t^3} - 2t - 4t^3 - 4t + 4t^5 + \cancel{4t^3}}{(1+t^2)^4}$$

$$= \frac{2t(t^4 - 2t^2 - 3)}{(1+t^2)^4}$$

This $=0$ if the numerator does:

$2t\,(t^4-2t^2-3)=0$

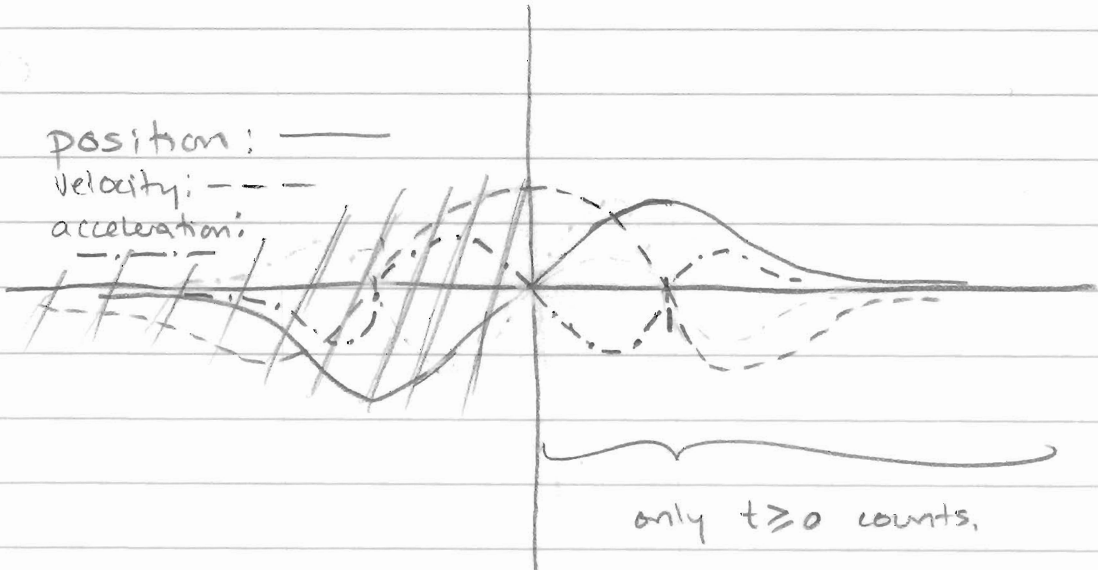
This happens if $t=0$ or if $(t^4-2t^2-3)=0$.
Let $u=t^2$. Then the second equation is
$3u^2-2u-3=0$

$u=\frac{2\pm\sqrt{4+4\cdot 3}}{6}=\frac{2\pm 4}{6}=1$ or $-\frac{1}{3}$ so $t^2=1$ if $t=\pm 1$

$t^2=-\frac{1}{3}$ nev

So the acceleration is 0 if $t=0,1,-1$;
but $t\geq 0$, so we only get $t=0$ or $t=1$ as solutions.

b) Use a graphing calculator for this:



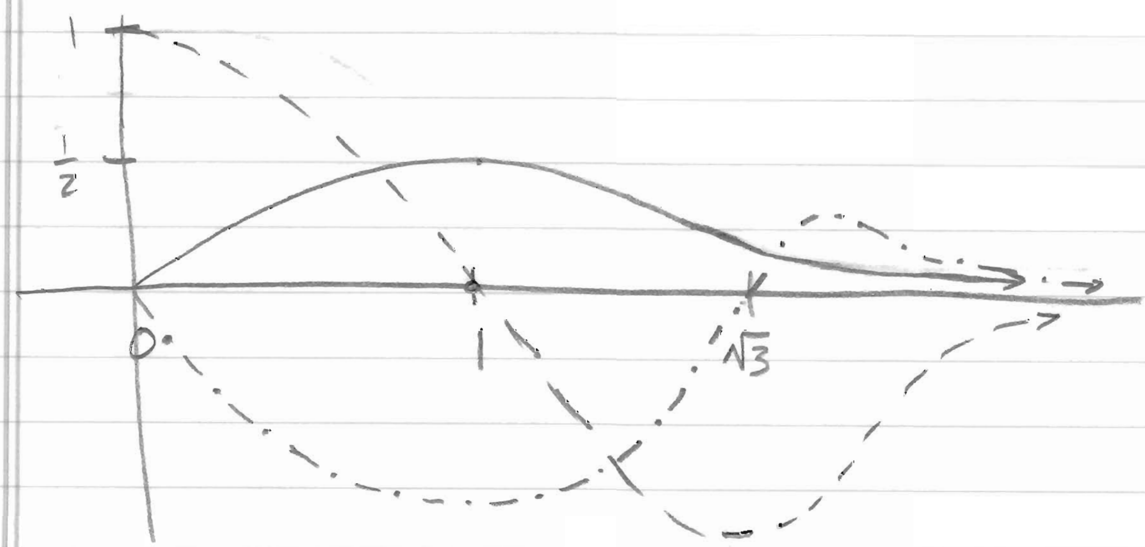
c) The particle starts out heading right.
Its velocity is positive, but decreasing. Thus it is slowing down (velocity getting closer to 0)
At $t=1$, its velocity is 0. Then it turns around and heads the other way.

$2t(t^4-2t^2-3)=0.$

This happens if $t=0$ or if $(t^4-2t^2-3)=0$.
Let $u=t^2$. Then the second equation becomes
$u^2-2u-3=0$, ie $(u-3)(u+1)=0$. So
$u=3$ or -1. so $t^2=3$ if $t=\pm\sqrt{3}$
$t^2=-1$ never.
So the possible t are 0 and $\sqrt{3}$, since the problem specifies $t \geq 0$.

So the acceleration is 0 if $t=0$ or $\sqrt{3}$.

b) Use a graphing calculator for this.



———— $= x(t)$

– – – – $= x'(t)$

–·–·–·– $= x''(t)$

c) The particle is travelling forward (right) for $0 \leq t \leq 1$. It is slowing down this whole time. (Its velocity is getting closer to 0)
The particle is travelling backwards (left) for $1 \leq t < \infty$. At first it is speeding up (its velocity is getting farther from 0), but after $t = \sqrt{3}$, it starts slowing down (its velocity begins to come back towards 0).

We see: If velocity > 0 and acc > 0 <u>or</u> velocity < 0 and acc < 0 } speeding up

if velocity > 0 and acc < 0 <u>or</u> velocity < 0 and acc > 0 } slowing down

ie, <u>Speeding up</u> if velocity & acceleration have the <u>same sign</u>
<u>Slowing down</u> if they have <u>opposite signs</u>.

3.7:66

a) If $F(x) = f(x)g(x)$ where f and g have derivatives of all orders, then

$$F''(x) = \frac{d^2}{dx^2}[f(x)g(x)] = \frac{d}{dx}\left[\frac{d}{dx}(f(x)g(x))\right]$$

$$= \frac{d}{dx}\left[\frac{d}{dx}[f(x)]g(x) + f(x)\frac{d}{dx}[g(x)]\right]$$

$$= \frac{d}{dx}\frac{d}{dx}[f(x)]g(x) + \frac{d}{dx}[f(x)]\frac{d}{dx}[g(x)]$$

$$+ \frac{d}{dx}[f(x)]\frac{d}{dx}[g(x)] + f(x)\frac{d}{dx}\left[\frac{d}{dx}[g(x)]\right]$$

$$= f''(x)g(x) + f'(x)g'(x) + f'(x)g'(x) + f(x)g''(x)$$

$$= f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)$$

b) $$\frac{d^3}{dx^3}[f(x)g(x)] = \frac{d}{dx}[f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x)]$$

$$= \frac{d}{dx}[f''(x)g(x)] + 2\frac{d}{dx}[f'(x)g'(x)] + \frac{d}{dx}[f(x)g''(x)]$$

$$= \left(\frac{d}{dx}[f''(x)]g(x) + f''(x)\frac{d}{dx}[g(x)]\right) + 2\left[\frac{d}{dx}[f'(x)]g'(x) + f'(x)\frac{d}{dx}[g'(x)]\right.$$
$$\left. + \left[\frac{d}{dx}[f(x)]g''(x) + f(x)\frac{d}{dx}[g''(x)]\right]\right)$$

$$= f'''(x)g(x) + f''(x)g'(x) + 2\left(f''(x)g'(x) + 2f'(x)g''(x)\right)$$
$$+ f'(x)g''(x) + f(x)g'''(x)$$

$$= f^{(3)}(x)g(x) + 3f^{(2)}(x)g'(x) + 3f'(x)g^{(2)}(x) + f(x)g^{(3)}(x)$$

c) Notice this is the same form as for the binomial $(x+y)^3 = x^3y^0 + 3x^2y + 3xy^2 + x^0y^3$

but where powers are replaced by derivatives.

So we guess $F^{(n)}$ will look like
$(x+y)^n$ but with
- x replaced by f
- y replaced by g
- powers replaced by derivatives

$n=0$						1				
$n=1$					1		1			
$n=2$				1		2		1		
$n=3$			1		3		3		1	
$n=4$		1		4		6		4		1

eg,
$$F^{(4)}(x) = f^{(4)}(x)g(x) + 4f^{(3)}(x)g'(x) + 6f^{(2)}(x)g^{(2)}(x) + 4f'(x)g^{(3)}(x) + f(x)g^{(4)}(x).$$

This turns out to be correct.