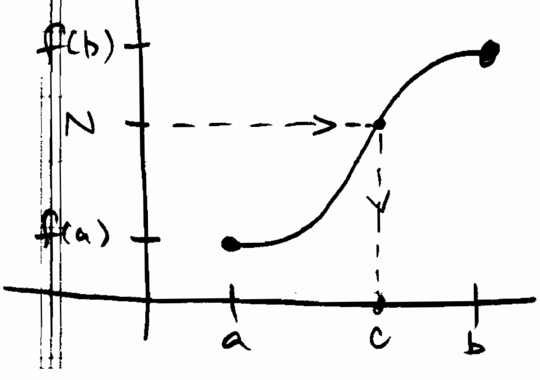
Midterm #1, Fall 2005
SOLUTIONS

1) Def A rational function $f(x)$ is a ratio of two polynomials $f(x) = \frac{p(x)}{q(x)}$, where $q(x) \neq 0$.

2) Def A function $f(x)$ is continuous on an open interval, (a,b), if it is continuous at x for all $x \in (a,b)$. It is continuous on a closed interval, $[a,b]$, if it is continuous on the open interval (a,b) and in addition satisfies $\lim_{x \to a^+} f(x) = f(a)$ and $\lim_{x \to b^-} f(x) = f(b)$.

3) Squeeze Theorem If $f(x) \leq g(x) \leq h(x)$ for all x near but not necessarily equal to a, and if $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$, then also $\lim_{x \to a} g(x) = L$.

4) Intermediate Value Theorem If $f(x)$ is continuous on the closed interval $[a,b]$ and if N is a number between $f(a)$ and $f(b)$ (where $f(a) \neq f(b)$) then there exists some $c \in (a,b)$ such that $f(c) = N$.



This says if f is continuous on $[a,b]$ then its graph must pass through every y value, N between $f(a)$ and $f(b)$.

5) T/F

a) True - this is a theorem

b) False - $f(x)$ may reach L, it just doesn't have to.

c) False - this is only an informal idea. Some continuous functions can't be drawn at all.

d) True - this is a theorem. They must also be equal.

e) False - eg, if $f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$ and $g(x) = \begin{cases} 0 & x \geq 0 \\ 1 & x < 0 \end{cases}$

then $f+g$ has a limit at 0 although f and g do not.

6) a) If $200 = P(V) = \dfrac{2 \cdot 8.314 \cdot 273.15}{V}$

then $V = \dfrac{2 \cdot 8.314 \cdot 273.15}{200} \approx 22.7097$ litres

b) If $205 = P(V_{upper})$

$V_{upper} = \dfrac{2 \cdot 8.314 \cdot 273.15}{205} \approx 22.1558$ litres

$V - V_{upper} \approx .5823$

$V_{lower} = \dfrac{2 \cdot 8.314 \cdot 273.15}{195} \approx 23.292$ litres

$V_{lower} - V \approx .553895$

we choose the smaller number as our error, so acceptable volume error is $\pm .553895$

c) a is 22.7097, L is 200, ε is 5, δ is .553895

7) a) $\lim_{t \to \infty} \sqrt{t} = \infty$, that is, as t increases to infinity, so does $f(t)$.

b) $f'(10)$ is the rate of seepage after 10 hours, in feet/hour.

c) $\lim_{t \to \infty} \frac{1}{2\sqrt{t}} = 0$, ie, as t increases to ∞, $f'(t)$ goes to 0.

d) The longer water seeps in, the deeper it goes. I'm not sure depth $\to \infty$ makes sense, but if we just think of ∞ as "very large" it makes sense that water can penetrate very deep over long periods. As it penetrates, either the soil may become denser, making it seep slower and slower, or the weight of the water decreases as some is held up by soil above, so it doesn't have the force to penetrate as fast, thus it slows to 0.

8) $f(x) = g \circ h(x)$ where $h(x) = x^3 + 3$ and $g(x) = \sqrt{x}$

9) $\lim_{x \to 2} \frac{4x^2 - 16}{x - 2} = \lim_{x \to 2} \frac{4(x^2 - 4)}{x - 2} = 4 \lim_{x \to 2} \frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}$

$= 4 \lim_{x \to 2} (x+2) = 4 \cdot (2+2) = 16$

10) We know $2x-3$ and x^2+c are polynomials, hence continuous, So

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2+c = 1^2+c = 1+c$$

and $$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x-3 = 2\cdot 1-3 = -1$$

also, $f(1) = 1^2+c = 1+c$, so we need these to be equal,

$$1+c = -1 \Rightarrow \boxed{c=-2}$$

So $f(x)$ is continuous at $x=1$ if $c=-2$.

11) $$f'(t) = \lim_{h\to 0} \frac{f(t+h)-f(t)}{h}$$

$$= \lim_{h\to 0} \frac{(2(t+h)^2-1)-(2t^2-1)}{h}$$

$$= \lim_{h\to 0} \frac{2(t^2+2th+h^2)\cancel{-1}-2t^2\cancel{+1}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{2t^2}+4th+2h^2-\cancel{2t^2}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(4t+2h)}{\cancel{h}} = \lim_{h\to 0} 4t+2h = 4t$$

So $f'(t) = 4t$.