MIDTERM #2 SOLUTIONS

1) A function $f(x)$ is said to be increasing if $a \le b$ implies $f(a) \le f(b)$. It is said to be decreasing if $a \le b$ implies $f(a) \ge f(b)$.

2a) Let f be a function defined at least on the interval (a,b). If f takes on an extreme value at a number c in this interval and if $f'(c)$ exists, then $f'(c)=0$.

b) $f(x) = x^{2/3}$ is defined on all of $\mathbb{R}$, hence on $(-4,4)$.
$f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$ is defined everywhere except at $x=0$.
$f'(x) \neq 0$ anywhere on $(-4,4)$, so the only point at which an extremum could occur is at $(0,0)$, which is in fact the minimum.

$f(x) = \sin(x^2)$ is defined on $(-4,4)$.
$f'(x) = \cos(x^2)\cdot 2x$ is also defined everywhere on $(-4,4)$, so the extrema, if they exist, must occur where $f'(x)=0$.
$0 = f'(x) = \cos(x^2)\cdot 2x$ when $x=0$ or when $\cos(x^2)=0$.
$\cos(\theta)=0$ when $\theta = \frac{\pi}{2}+n\pi$, so when $x = \sqrt{\frac{\pi}{2}+n\pi}$
$(n \in \{-3,-2,-1,0,1,2\})$ these are minima for n : and maxima for n even

3) Graph $f(x) = \frac{x}{x^2+1}$:

intercepts: x-intercept = y-intercept = (0,0)
no vertical asymptotes
horizontal asymptote at 0 as $x \to \pm\infty$

$$f'(x) = \frac{(x^2+1) - x \cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$ this is everywhere defined, so

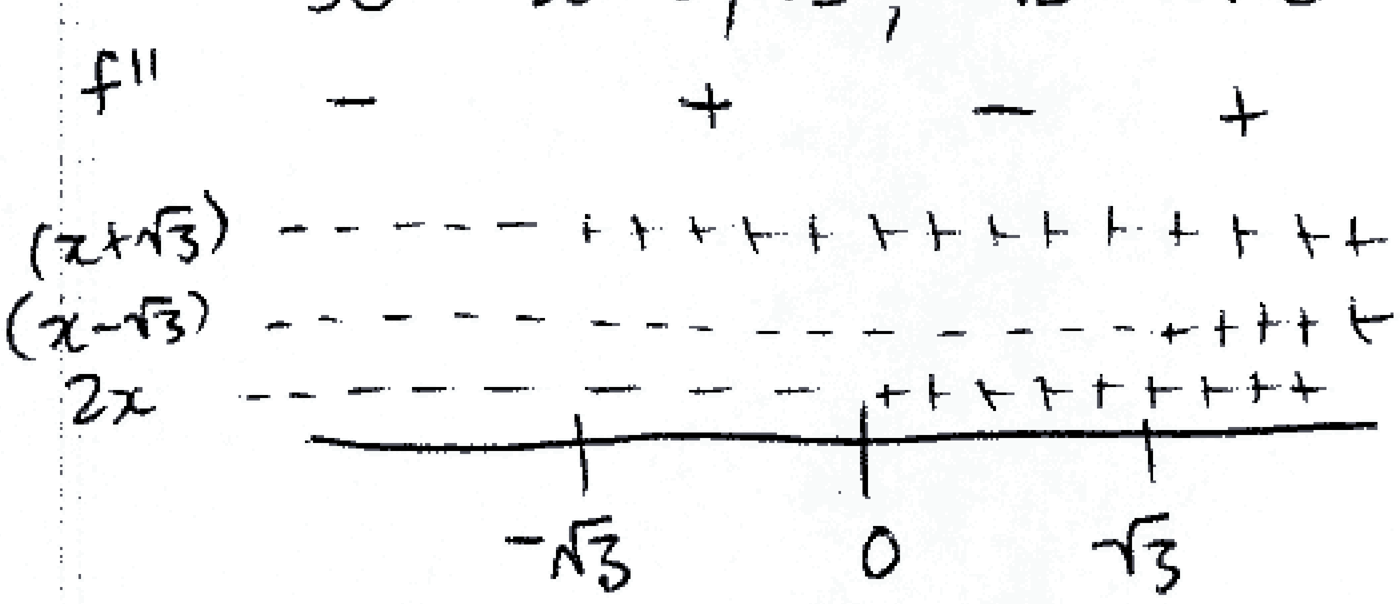
critical pts: where $f'(x) = 0$ are $\pm 1 = x$ $f(1) = \frac{1}{2}$, $f(-1) = \frac{-1}{2}$
f is increasing for $-1 < x < 1$, since $f'(x) > 0$ there
decreasing for $-\infty < x < -1$, $1 < x < \infty$, since $f'(x) < 0$ there

$$f''(x) = \frac{-2x(x^2+1)^2 - 2(1-x^2)(x^2+1) \cdot 2x}{(x^2+1)^4}$$ factor out an (x^2+1)

$$= \frac{-2x(x^2+1) - 2(1-x^2) \cdot 2x}{(x^2+1)^3} = \frac{-2x^3 - 2x - 4x + 4x^3}{(x^2+1)^3}$$

$$= \frac{2x^3 - 6x}{(x^2+1)^3} = \frac{2x(x^2-3)}{(x^2+1)^3}$$

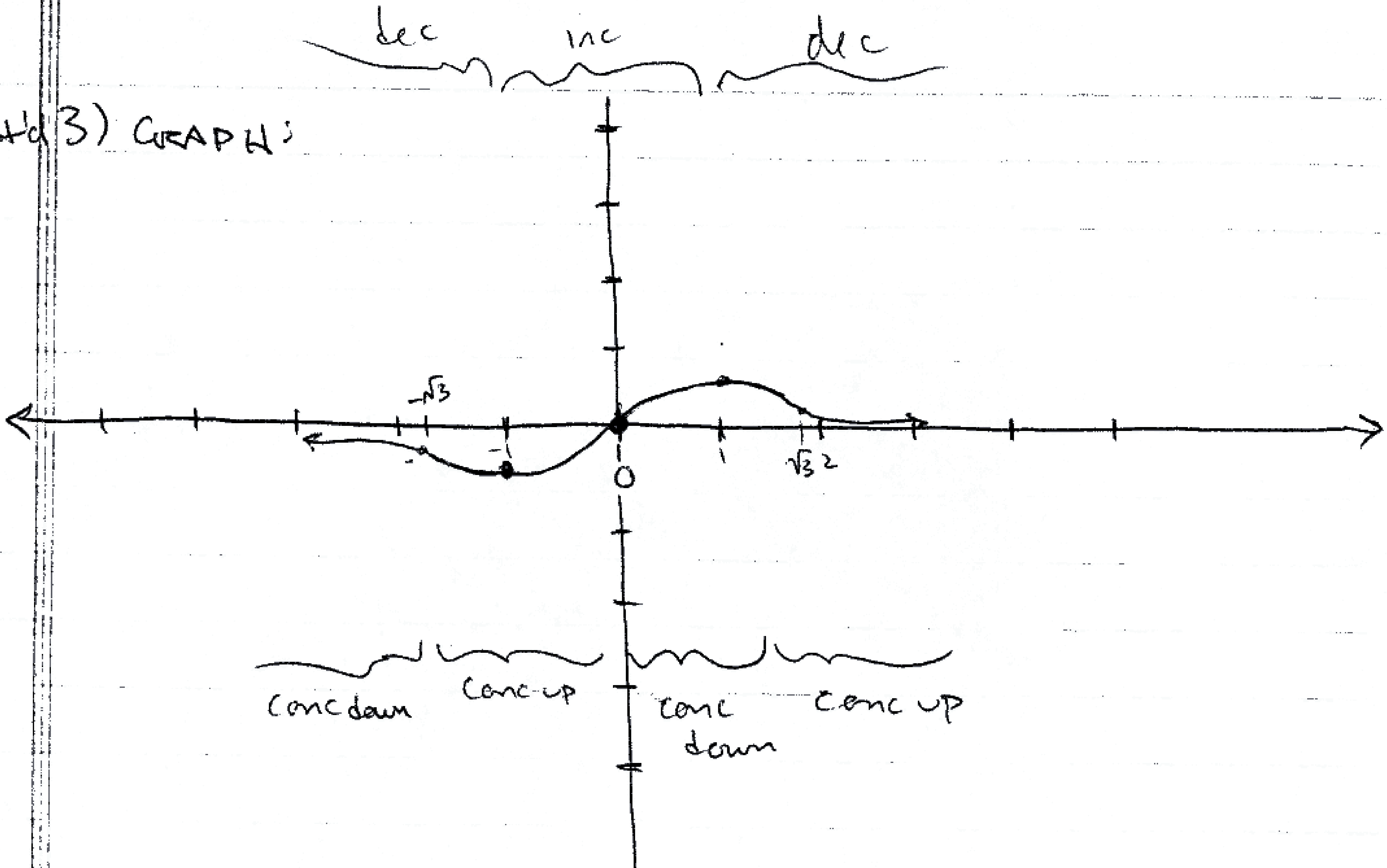
So $x = 0, \sqrt{3}, -\sqrt{3}$ are possible inflection points



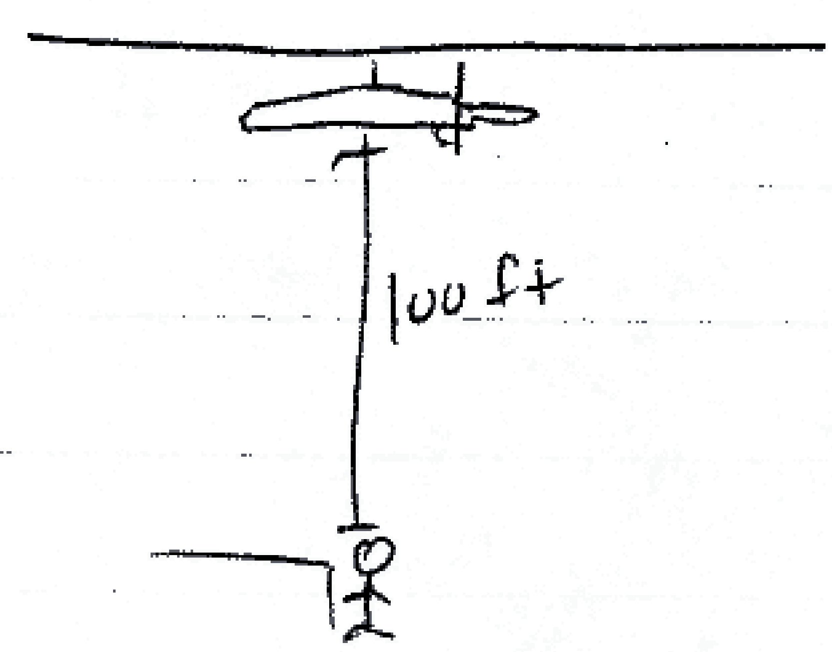
f is concave down on $(-\infty, -\sqrt{3})$ and $(0, \sqrt{3})$ and concave up on $(-\sqrt{3}, 0)$, $(\sqrt{3}, \infty)$.

So all three points are inflection points
$f(-\sqrt{3}) = \frac{-\sqrt{3}}{4}$, which is between 0 and −1 $f(\sqrt{3}) = \frac{\sqrt{3}}{4}$ which is between 0 and 1.

Cont'd 3) GRAPH:



4)



↙ acceleration in ft/sec² due to gravity

$$h(t) = \frac{-32}{2}t^2 + V_0 t + h_0$$

$$= \frac{-32}{2}t^2 + 0t + 100$$

$$= \frac{-32}{2}t^2 + 100$$

$h(t) = 0$ when it hits him, so this will occur at

$0 = \frac{-32}{2}t^2 + 100 \rightsquigarrow t = \sqrt{\frac{200}{32}} = \sqrt{\frac{100}{16}}$ seconds $= 2.5$ seconds

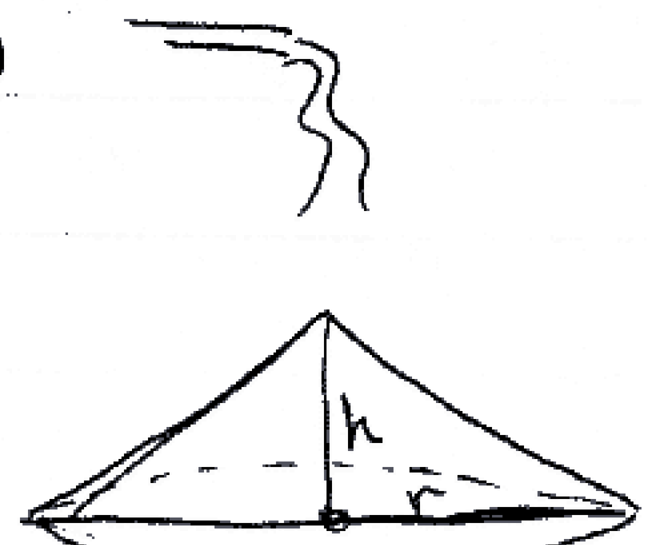
if he doesn't move quicker than that.

If he isn't quick enough, it will hit him at a velocity of $v(t) = h'(t) = -32t$ ft/sec

$$h'\left(\tfrac{5}{2}\right) = -32\tfrac{5}{2} \text{ ft/sec} = -80 \text{ ft/sec}$$

the speed is then $|v(t)| = 80$ ft/sec

5)



we're given:

$$h = \frac{1}{3} r \Rightarrow 3h = r$$

also $V = \frac{\pi r^2 h}{3}$, $\frac{dV}{dt} = 120\pi \ ft^3/sec$

we want $\frac{dh}{dt}$. Since $r = 3h$,

$$V = \frac{\pi (3h)^2 h}{3} = 3\pi h^3$$

implicitly differentiating wrt time:

$$\frac{dV}{dt} = 3\pi 3h^2 \frac{dh}{dt}$$

Solving for $\frac{dh}{dt}$:

$$\frac{dh}{dt} = \frac{\frac{dV}{dt}}{9\pi h^2} = \frac{\overset{40}{\cancel{120}}\cancel{\pi}}{{}_{3}\cancel{9}\cancel{\pi} h^2} = \frac{40}{3h^2}$$

So when $h = 20$, $\frac{dh}{dt} = \frac{40}{3(20)^2} = \frac{2}{3 \cdot 20} = \frac{1}{30}$ ft/sec

(6) $\frac{1}{x+1} + \frac{1}{y+1} = 1$ implies

$$\frac{d}{dx}\left(\frac{1}{x+1} + \frac{1}{y+1}\right) = \frac{d}{dx}(1) = 0$$

$$\Rightarrow \frac{d}{dx}\left((x+1)^{-1}\right) + \frac{d}{dx}\left((y+1)^{-1}\right) = 0$$

$$-(x+1)^{-2} \cdot 1 + \frac{dy}{dx} \Big/ -(y+1)^2 = 0$$

$$\frac{dy}{dx} \Big/ -(y+1)^2 = \frac{1}{(x+1)^2}$$

$$\frac{dy}{dx} = \frac{-(y+1)^2}{(x+1)^2}$$

→ let $u = (v)^{-1}$

$v = y+1$

then $\frac{du}{dx} = \frac{du}{dv}\frac{dv}{dy}\frac{dy}{dx}$

$= -v^{-2} \cdot 1 \cdot \frac{dy}{dx}$

$= \frac{\frac{dy}{dx}}{-(y+1)^2}$

so at (1,1),

$$= \frac{-(1+1)^2}{(1+1)^2} = -1$$

Then using point-slope:

$(y-1) = -1(x-1)$ or $y = -x+2$

extra credit

If we let $f(t)$ be how far the car has travelled by time t, starting with noon at $t=0$, then we can assume by physics that $f(t)$ is continuous and $f'(t)$ exists everywhere on $\mathbb{R}$, hence on $[0,1]$.

So we can apply the mean value theorem to get

$$\frac{f(1)-f(0)}{1-0} = f'(c) \quad \text{for some } c \in [0,1]$$

$$\frac{12407-12337}{1} = f'(c)$$

So $f'(c) = 70$ mph at some $c \in [0,1]$.

Since velocity is $f'(c)$, this means at some time c between noon and 1:00 the car was going 70 mph, which is speeding.