PRACTICE MIDTERM #2 SOLUTIONS

1) The function f has a global maximum (resp, minimum) at c if $f(c) \geq f(x)$ (resp, $f(c) \leq f(x)$) for all x in the domain of f.

2) a) The Mean Value Theorem: If $f(x)$ is continuous on the closed interval $[a,b]$ and differentiable on (a,b), then there exists a point $c \in (a,b)$ with

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

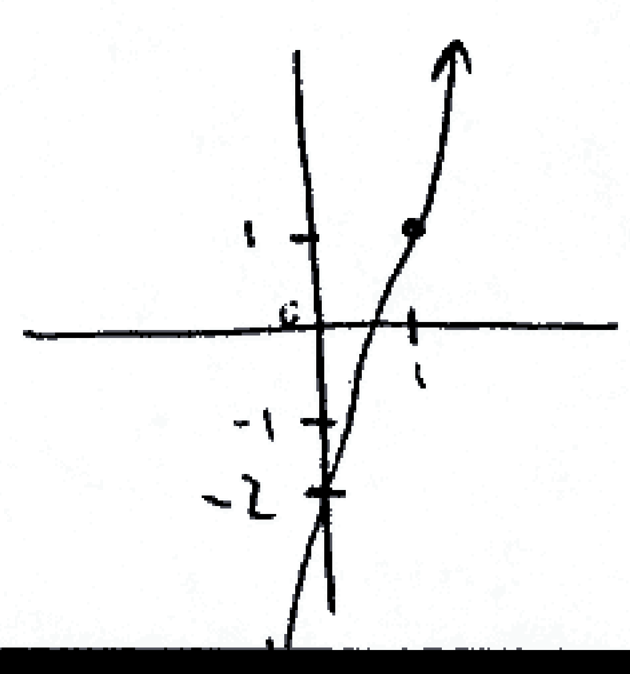
b) $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$, which is undefined at $x=0$ so we can't apply the theorem

$$\frac{f(8)-f(-1)}{8-(-1)} = \frac{4-1}{9} = \frac{3}{9} = \frac{1}{3}$$

$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} = \frac{1}{3}$ when $x^{-\frac{1}{3}} = \frac{1}{2} \Rightarrow x = \left(\frac{1}{2}\right)^{-3} = 8 \notin (-1,8)$.

so $c = \nexists$ does not satisfy the conclusion of the theorem.

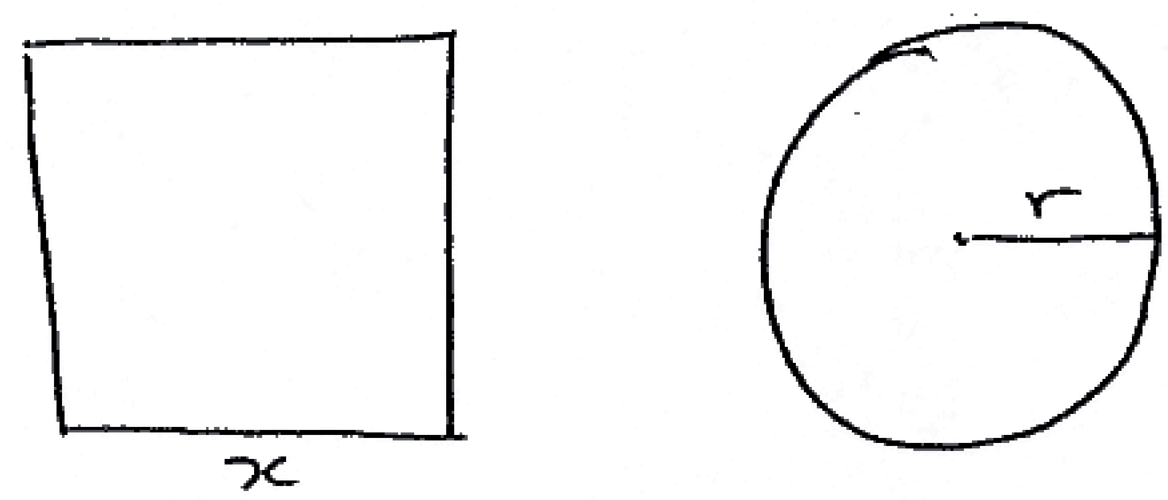
3) $f(x) = x^5 + 2x^3 - 2$ has y-intercept -2, and also passes through the point $(1,1)$. $f'(x) = 5x^4 + 6x^2 = x^2(5x^2+6)$ So 0 is the only possible critical point. $f(x)$ is increasing on all of $\mathbb{R}$.



Already, this is enough to show there is a root and only one.

The intermediate value theorem implies there is a root, and f increasing implies it is unique.

4)



$25 \geq x \geq 0$

total Circumference $= 4x + 2\pi r = 100$ ft is the constraint. $0 \leq r \leq \frac{100}{2\pi}$ is the inter

$x^2 + \pi r^2 = \text{Area}$ is what we want to minimize.

Solve for x in terms of r

$$x = \frac{100 - 2\pi r}{4} = 25 - \frac{\pi}{2} r$$

plug into Area:

$$A(r) = \left(25 - \frac{\pi}{2} r\right)^2 + \pi r^2 = 625 - 25\pi r + \frac{\pi^2}{4} r^2 + \pi r^2$$

$$= \left(\frac{\pi^2}{4} + \pi\right) r^2 - 25\pi r + 625$$

Critical points are 1) endpoints $0, \frac{100}{2\pi}$

2) where $A'(r)$ doesn't exist (none)

3) where $A'(r) = 0$:

$$0 = A'(r) = 2\left(\frac{\pi^2}{4} + \pi\right) r - 25\pi$$

$$0 = \left(\frac{\pi}{2} + 2\right) r - 25$$

$$\frac{25}{\frac{\pi}{2} + 2} = r = \frac{50}{4 + \pi}$$

evaluate A at critical pts:

$$A(0) = 25^2 = 625$$

$$A\left(\frac{100}{2\pi}\right) = \left(25 - \frac{\pi}{2} \cdot \frac{100}{2\pi}\right)^2 + \pi \left(\frac{100}{2\pi}\right)^2$$

$$= \frac{50^2}{\pi} = \frac{2500}{\pi}$$

$$A\left(\frac{50}{4+\pi}\right) = \left(25 - \frac{\pi}{2} \cdot \frac{50}{4+\pi}\right)^2 + \pi \left(\frac{50}{4+\pi}\right)^2$$

$$= \left(25 - \frac{25\pi}{4+\pi}\right)^2 + \pi \frac{50^2}{(4+\pi)^2}$$

4 cont'd) Since $\pi \sim 3$,

$$A\left(\frac{100}{2\pi}\right) \sim \frac{2500}{3} \sim 633$$

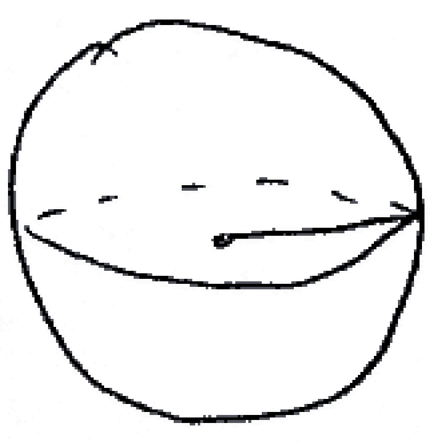
$$A\left(\frac{50}{4\pi}\right) = \left(25 - \frac{25\pi}{4+\pi}\right)^2 + \pi \frac{}{(4+\pi)^2} \sim \left(25 - \frac{75}{7}\right)^2 + \frac{3}{49}.$$

$$\sim (25 - 10.5)^2 + 3 \cdot 50 =$$

$$(14.5)^2 + 150 = 100.25 + 150 = 350.25$$

So the minimum will be at $\frac{50}{4+\pi}$, about 350 square feet of space.

(even without a calculator, you can figure which one must be the minimum - there is no need to calculate precisely)

5) 

$$\frac{4}{3}\pi r^3 = \text{Volume}$$

Implicit differentiate both sides with respect to t:

$$\frac{4}{3}\pi\, 3r^2 \cdot \frac{dr}{dt} = \frac{dV}{dt} = 50 \text{ in}^3/\text{min}$$

$$\text{So } \frac{dr}{dt} = \frac{50}{4\pi r^2}$$

So at $r = 5$, $\frac{dr}{dt} = \frac{50}{4 \cdot \pi \cdot 25} = \frac{1}{2\pi}$ in/min

6) Implicit differentiate

$\sin(x+y) = y$ with respect to x!

$$\frac{d}{dx}(\sin(x+y)) = \frac{dy}{dx}$$

$u = \sin v$ $\quad \frac{du}{dx} = \frac{du}{dv} \cdot \frac{dv}{dx} = \cos v \cdot \left(1 + \frac{dy}{dx}\right) = \cos(x+y)\left(\frac{dy}{dx} + 1\right)$

$v = x+y$

$$\text{So } \cos(x+y)\left(\frac{dy}{dx} + 1\right) = \frac{dy}{dx}.$$

6) cont'd)

Now solve for $\frac{dy}{dx}$ in terms of x and y:

$$\frac{dy}{dx}\cos(x+y) + \cos(x+y) = -\frac{dy}{dx}$$

$$\frac{dy}{dx}\cos(x+y) + \frac{dy}{dx} = -\cos(x+y)$$

$$\frac{dy}{dx}\left(\cos(x+y)+1\right) = -\cos(x+y)$$

$$\frac{dy}{dx} = \frac{-\cos(x+y)}{\cos(x+y)+1}$$

plug in $(0,0)$:

$$\frac{dy}{dx} = \frac{-\cos(0+0)}{\cos(0+0)+1} = \frac{-1}{2}$$

so we have a point, $(0,0)$ and a slope $= \frac{-1}{2}$, so the tangent line is

$$y = \frac{-1}{2}x. \qquad (y\text{-intercept} = 0)$$