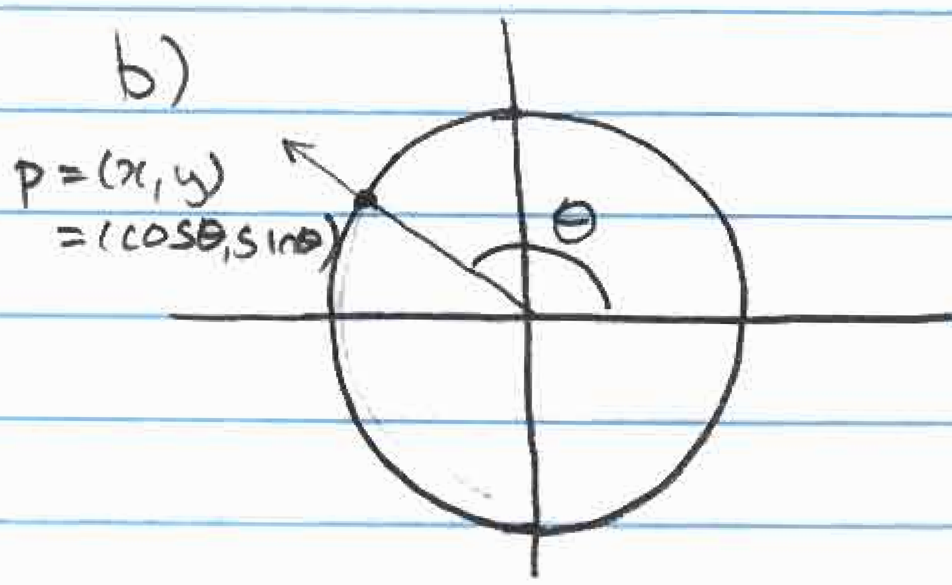
CALCULUS 140
MIDTERM #2
FALL 2003
SOLUTIONS

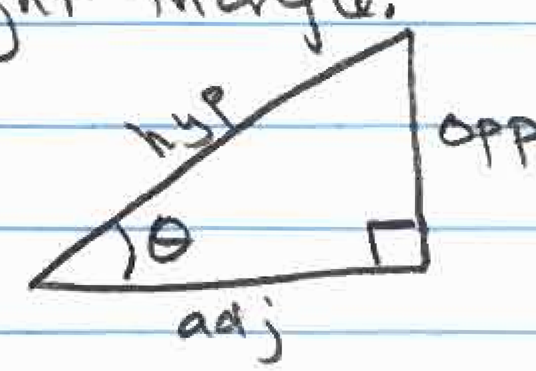
1) a) <u>Def</u> Let $p=(x,y)$ be the point on the unit circle where it intersects the ray which is rotated an angle of θ radians counterclockwise from the positive x-axis. Then we define $\cos(\theta)=x$ and $\sin(\theta)=y$.

b)



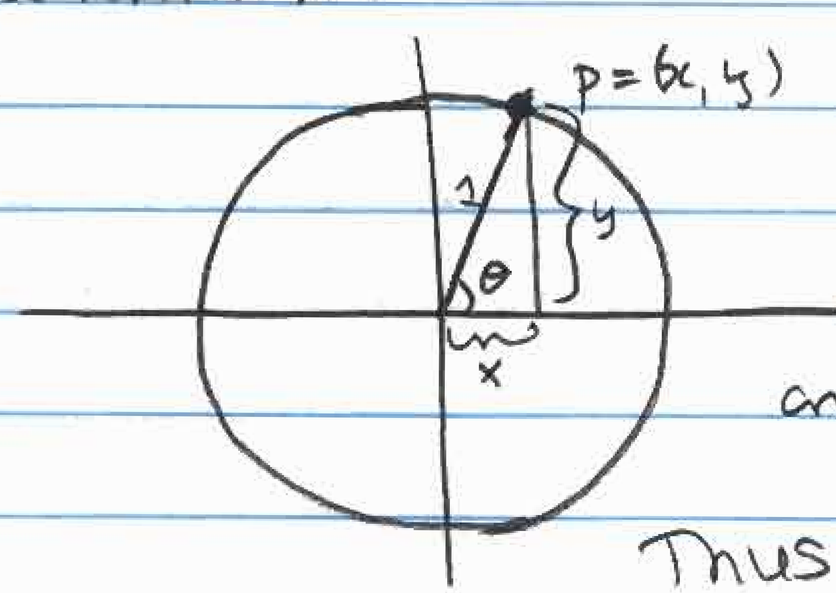
The original definitions are:
Let θ be an interior angle other than the right angle in a right triangle.

Then $\sin\theta = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse}}$:



$\cos\theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$

This definition only makes sense for $\theta \in (0, \frac{\pi}{2})$. So we must check that the unit circle definition gives the same values for $\theta \in (0, \pi/2)$ as the triangle definition:



By old def, $\sin\theta = \frac{y}{1} = y$ = new def of $\sin\theta$

and old def $\cos\theta = \frac{x}{1} = x$ = new def of $\sin\theta$.

Thus those definitions agree.

2) <u>Thm</u> If $f(x)$ is differentiable at x and if $f(x) \neq 0$, then $\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{-\frac{d}{dx}(f(x))}{[f(x)]^2}$.

If $f(x) = x^2$, then $\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx}\left(x^{-2}\right)$
$= -2x^{-3} = \frac{-2}{x^3}$.

Whereas, $\frac{1}{\frac{d}{dx}[f(x)]} = \frac{1}{\frac{d}{dx}[x^2]} = \frac{1}{2x}$

If, eg, $x = 1$, $\frac{-2}{1^3} = -2 \neq \frac{1}{2} = \frac{1}{2 \cdot 1}$

So $\frac{d}{dx}\left[\frac{1}{f(x)}\right] \neq \frac{1}{\frac{d}{dx}[f(x)]}$.

Thus we see that the derivative of the reciprocal of a function is <u>not</u> equal to the reciprocal of its derivative.

3) a) <u>Def</u> Two functions $f(x)$ and $g(y)$ are called <u>inverse functions</u> if $f(g(y)) = y$ for all y in the domain of g and $g(f(x)) = x$ for all x in the domain of f.

b) <u>Thm</u> If f and g are inverse functions and if f is differentiable at $g(y)$ and $f'(g(y)) \neq 0$, then g is differentiable at y and
$$g'(y) = \frac{1}{f'(g(y))}.$$

4) a) $\frac{d}{dx}[\ln(3x^2)] = \frac{d}{dx}[\ln 3 + \ln x^2]$

$= \frac{d}{dx}[\ln 3 + 2\ln x]$

$= \frac{d}{dx}[\ln 3] + 2\frac{d}{dx}[\ln x]$

$= 0 + \frac{2}{x} = \frac{2}{x}.$

Alternatively, $\frac{d}{dx}[\ln(3x^2)] = \frac{\frac{d}{dx}[3x^2]}{3x^2}$ by chain rule for ln.

$= \frac{6x}{3x^2} = \frac{2}{x}.$

b) $\frac{d}{dx}\left[\frac{e^{3x}+2}{x^2+5x-2}\right] = \frac{d}{dx}\left[\frac{f}{g}\right]$, where

$f(x) = e^{3x}+2$ $\qquad$ $g(x) = x^2+5x-2$

$\frac{d}{dx}[f(x)] = 3e^{3x}$ $\qquad$ $\frac{d}{dx}[g(x)] = 2x+5$

So $\frac{d}{dx}\left[\frac{f}{g}\right] = \frac{\frac{d}{dx}[f]\cdot g - f\cdot\frac{d}{dx}[g]}{g^2}$

$= \frac{3e^{3x}[x^2+5x-2] - [e^{3x}+2][2x+5]}{(x^2+5x-2)^2}$

$= \frac{3x^2e^{3x} + 15xe^{3x} - 6e^{3x} - 2xe^{3x} - 4x - 5e^{3x} - 10}{(x^2+5x-2)^2}$

$= \frac{e^{3x}(3x^2+13x-11) - 4x - 10}{(x^2+5x-2)^2}$

c) $\frac{d}{dx}[(\arctan x)^2]$ $\qquad$ $y = \arctan x$ $\qquad$ $\frac{dy}{dx} = \frac{1}{1+x^2}$

$= \frac{d}{dx}[y^2]$ $\qquad$ $v = y^2$ $\qquad$ $\frac{dv}{dy} = 2y$

$= \frac{d}{dx}[v] = \frac{dv}{dx} = \frac{dv}{dy}\cdot\frac{dy}{dx} = 2y\cdot\frac{1}{1+x^2} = \frac{2\arctan x}{1+x^2}.$

5) a) $\frac{d}{dx}[y^3+xy] = \frac{d}{dx}[x^2+1]$

$$\frac{d}{dx}[y^3] + \frac{d}{dx}[xy] = \frac{d}{dx}[x^2] + \frac{d}{dx}[1]$$

$$3y^2\frac{dy}{dx} + \frac{d}{dx}[x]y + x\frac{d}{dx}[y] = 2x + 0$$ by the product rule

$$3y^2\frac{dy}{dx} + y + x\frac{dy}{dx} = 2x$$

$$3y^2\frac{dy}{dx} + x\frac{dy}{dx} = 2x - y$$

$$(3y^2+x)\frac{dy}{dx} = 2x - y$$

$$\frac{dy}{dx} = \frac{2x-y}{3y^2+x}$$

b) at (1,1), $\frac{dy}{dx} = \frac{2\cdot 1 - 1}{3\cdot 1^2+1} = \frac{2-1}{3+1} = \frac{1}{4}$

So by point-slope form,

$$y - 1 = \frac{1}{4}(x-1) = \frac{1}{4}x - \frac{1}{4}$$

So $y = \frac{1}{4}x + \frac{3}{4}$ is the tangent line to this curve at the point (1,1).

6) a) $\frac{d}{dT}[PV] = \frac{d}{dT}[nRT]$ V, n, R constant, so

$$V\frac{d}{dT}[P] = nR\frac{d}{dT}[T]$$

$$V\frac{dP}{dT} = nR$$

$$\frac{dP}{dT} = \frac{nR}{V}$$

This is the approximate increase in atmospheres of pressure of the gas per degree kelvin increase in temperature.

b) If $n = .03$, $V = .5$, $T = 300K$, then

$$P = \frac{nRT}{V} = \frac{.03 \cdot 0.0821 \cdot 300}{.5} \approx 1.4778 \text{ atmospheres}$$

$$\frac{dP}{dT} = \frac{nR}{V} = \frac{.03 \cdot 0.0821}{.5} \approx .004926 \ \frac{\text{atmospheres}}{\text{degree kelvin}}$$

so

Pressure at 302K is about

$\approx$ pressure at 300K + $\frac{dP}{dT} \cdot \Delta T$

$$= 1.4778 + .004926 \cdot 2 \approx 1.48765 \text{ atmospheres}$$

7) a) $v(t) = h'(t) = \frac{d}{dt}\left[-4e^{-.05t}\cos(t) + 5\right]$

$$= -4\frac{d}{dt}\left[e^{-.05t}\cos t\right] + \cancel{\frac{d}{dt}[5]}^{\,0}$$

$$= -4\left(\frac{d}{dt}\left[e^{-.05t}\right]\cos t + e^{-.05t}\frac{d}{dt}[\cos t]\right)$$

$$= -4\left(-.05e^{-.05t}\cos t - e^{-.05t}\sin t\right)$$

$$= .2e^{-.05t}\cos t + 4e^{-.05t}\sin t$$

b) $a(t) = v'(t) = h''(t) = \frac{d}{dt}\left[.2e^{-.05t}\cos t + 4e^{-.05t}\sin t\right]$

$$= .2\left(-.05e^{-.05t}\cos t - e^{-.05t}\sin t\right)$$

$$+ 4\left(-.05e^{-.05t}\sin t + e^{-.05t}\cos t\right)$$

$$= -.01e^{-.05t}\cos t - 0.2e^{-.05t}\sin t - .2e^{-.05t}\sin t + 4e^{-.05t}\cos t$$

$$= 3.99e^{-.05t}\cos t - 0.4e^{-.05t}\sin t$$

c) $j(t) = a'(t) = v''(t) = h'''(t) = \frac{d}{dt}\left[3.99e^{-.05t}\cos t - 0.4e^{-.05t}\sin t\right]$

$$= 3.99\left(-.05e^{-.05t}\cos t - e^{-.05t}\sin t\right)$$

$$-.4\left(-.05e^{-.05t}\sin t + e^{-.05}\cos t\right)$$

$$= -.1995e^{-.05t}\cos t - 3.99e^{-.05t}\sin t + .02e^{-.05t}\sin t - .4e^{-.05t}\cos t$$

$$= -.5995e^{-.05t}\cos t - 3.97e^{-.05t}\sin t$$

d) $h(0) = -4e^{-.05\cdot 0}\cos(0) + 5 = -4\cdot 1\cdot 1 + 5 = 1$

$h(\frac{\pi}{2}) = -4e^{-.05\cdot\pi/2}\cos\frac{\pi}{2} + 5 = -4\cdot ?\cdot 0 + 5 = 5$

$V(0) = .2e^{-.05\cdot 0}\cos 0 + 4e^{-.05\cdot 0}\sin 0 = .2\cdot 1\cdot 1 + 4\cdot 1\cdot 0 = .2$

$V(\frac{\pi}{2}) = .2e^{-.05\cdot\frac{\pi}{2}}\cos\frac{\pi}{2} + 4e^{-.05\frac{\pi}{2}}\sin\frac{\pi}{2} = .2\cdot ?\cdot 0 + 4\cdot e^{-.05\frac{\pi}{2}}\cdot 1$

≈ 3.698

$a(0) = 3.99e^{-.05\cdot 0}\cos 0 - .4e^{-.05\cdot 0}\sin 0 = 3.99$

$a(\frac{\pi}{2}) = 3.99e^{-.05\frac{\pi}{2}}\cos\frac{\pi}{2} - .4e^{-.05\pi/2}\sin\frac{\pi}{2} = -.4e^{-.05\pi/2} \approx -.369786$

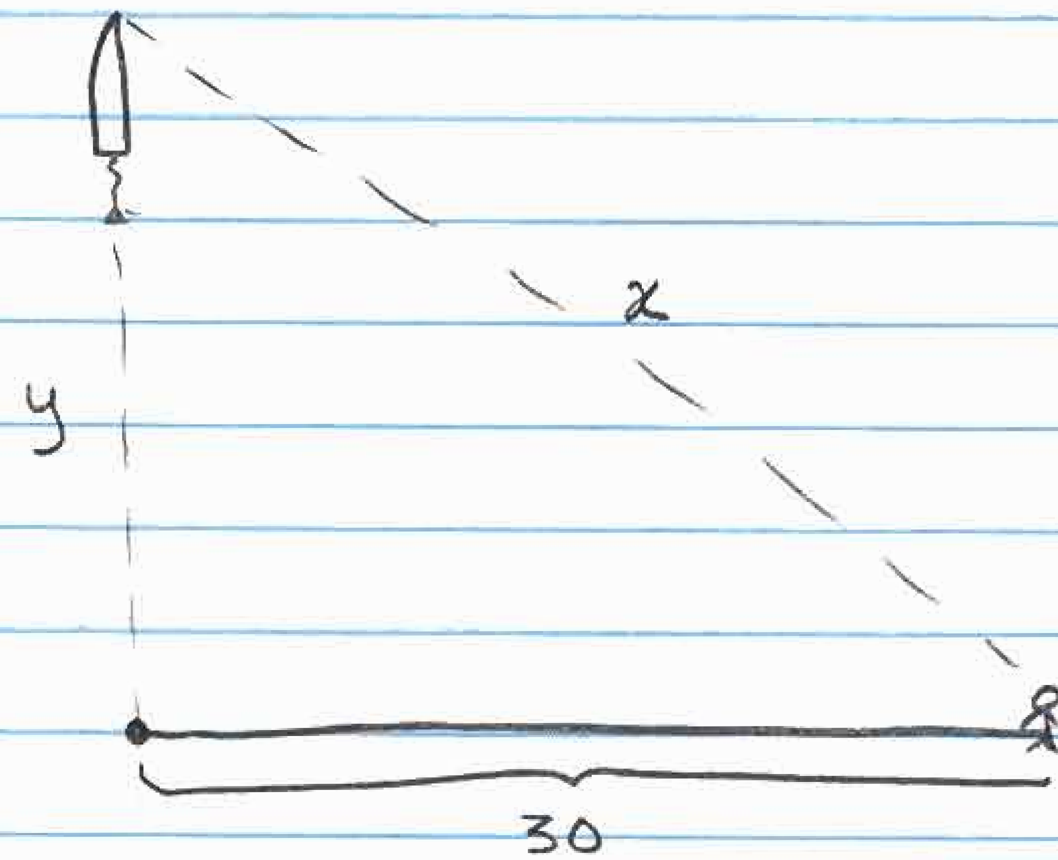
$j(0) = -.5995e^{-.05\cdot 0}\cos 0 - 3.97e^{-.05\cdot 0}\sin 0 = -.5995$

$j(\frac{\pi}{2}) = -.5995e^{-.05\cdot\pi/2}\cos\frac{\pi}{2} - 3.97e^{-.05\pi/2}\sin\pi/2 = -3.97e^{-.05\pi/2}$

≈ -3.67013

e) higher at $\frac{\pi}{2}$, faster at $\frac{\pi}{2}$, more pressure at 0, more whiplash at $\frac{\pi}{2}$ (since $|j(\frac{\pi}{2})| > |j(0)|$).

8)



When $y = 40$, $x = \sqrt{30^2 + 40^2} = \sqrt{900 + 1600} = \sqrt{2500} = 50$

and $\frac{dy}{dt} = 15$ ft/sec. We want $\frac{dx}{dt}$.

In general, $y^2 + 30^2 = x^2$

So $\frac{d}{dt}[y^2 + 30^2] = \frac{d}{dt}[x^2]$

$\frac{d}{dt}[y^2] + \frac{d}{dt}[900] = \frac{d}{dt}[x^2]$

$2y\frac{dy}{dt} + 0 = 2x\frac{dx}{dt}$

Thus when $y = 40$, $x = 50$, $\frac{dy}{dt} = 15$,

$2 \cdot 40 \cdot 15 = 2 \cdot 50 \cdot \frac{dx}{dt}$

$\frac{dx}{dt} = \frac{2 \cdot 40 \cdot 15}{2 \cdot 50} = 12$ ft/sec is the speed at which the distance between the observer and the Roman candle is increasing.