1 a) $\lim_{x \to a} f(x) = L$ if no matter how close you want $f(x)$ to be to L, you can get it that close (eg, within $\varepsilon > 0$) by choosing x sufficiently close to a (within δ, where δ is given in terms of ε).

(Any equivalent answer is fine)

b) $\lim_{x \to 2} 5x - 3 = 7$ if given any $\varepsilon > 0$, we can find δ in terms of ε such that if $0 < |x-2| < \delta$, then $|(5x-3) - 7| < \varepsilon$. Calculating backwards, we get

$$-\varepsilon < (5x-3) - 7 < \varepsilon \quad \text{if}$$

$$-\varepsilon < 5x - 10 < \varepsilon \quad \text{if}$$

$$-\frac{\varepsilon}{5} < x - 2 < \frac{\varepsilon}{5}.$$

So if x is within $\frac{\varepsilon}{5}$ of 2, then $(5x-3)$ is within ε of 7.

So $\delta = \frac{\varepsilon}{5}$ is what we needed to find.

2 a) To show $f(x)$ is continuous on $[-2, 2]$, it suffices to show f is continuous on a larger open interval, eg $\mathbb{R}$.

Claim $f(x) = x^5 - 3x + 4$ is continuous on $\mathbb{R}$

pf Let c denote an arbitrary element of $\mathbb{R}$.

1) $f(c) = c^5 - 3 \cdot c + 4$ is defined for all such c

2) $\lim_{x \to c} f(x) = \lim_{x \to c} x^5 - 3x + 4 = \lim_{x \to c} x^5 - \lim_{x \to c} 3x + \lim_{x \to c} 4$

by addition & subtraction rules

2a, cont'd) $= \left(\lim_{x\to c} x\right)^{\lim_{x\to c} 5} - 3\lim_{x\to c} x + \lim_{x\to c} 4$

by constant multiple rule

"on probation" - need either $\lim_{x\to c} x > 0$ or $\lim_{x\to c} 5 \in \mathbb{Z}$

$= (c)^5 - 3c + 4$ by rules 00, 0. Note since $\lim_{x\to c} 5 \in \mathbb{Z}$, we were justified in our last step.

3) these are equal, so f is continuous at c

Therefore, since c was arbitrary, f is continuous on $\mathbb{R}$. ∎

b) <u>The Intermediate Value Theorem</u> states that if $f(x)$ is continuous on the closed interval $[a,b]$ and if $f(a) < w < f(b)$ or $f(b) < w < f(a)$, then there is some $c \in [a,b]$ such that $f(c) = w$.

So to prove f has a root on $[-2,2]$, since we know it is continuous there, it suffices to show that 0 is in between $f(-2)$ and $f(2)$.

$f(-2) = (-2)^5 - 3(-2) + 4 = -32 + 6 + 4 = -22$

$f(2) = 32 - 3(2) + 4 = 32 - 6 + 4 = 30$

$-22 < 0 < 30$, so f must have a root in $[-2,2$

3) f(x) has no limit at: -2 because the function goes to $\infty, -\infty$ from the left and right.

: 0 since from the right, there are too many possible Limits for f

: 3 also because there are 2 different possible limits, so neither one is <u>the</u> limit.

f(x) is not continuous at: -2, 0, 3, since it does not have a limit at these points

: 5, since f(x) is not defined at 5

: 6, since $\lim_{x \to 6} f(x) \neq f(6)$

f(x) is not differentiable at: -2, 0, 3, 5, 6, since it is not continuous at these points

: -5, 7, since there are "corners" in the graph here => more than one possible tangent line so no single tangent line.

4) a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

b)

DO THIS BY YOURSELF. I WILL LOOK OVER YOUR ANSWER IF YOU LIKE.

c)
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{2x\cancel{h} + h^{\cancel{2}}}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h = 2x$$

So $f'(x) = 2x$

5) a) $\lim_{x\to 1} 7(2^{5x-4}) = 7 \lim_{x\to 1} (2^{5x-4})$ by constant multiple rule (5

$= 7 \left(\lim_{x\to 1} 2\right)^{\lim_{x\to 1} 5x-4}$ "on probation" using the exponent rule - need either

- $\lim_{x\to 1} 2 \geq 0$

or

- $\lim_{x\to 1} 5x-4 \in \mathbb{Z}$

$= 7(2)^{\left(\lim_{x\to 1} 5x - \lim_{x\to 1} 4\right)}$ by difference/sum rule

off probation from last step, since $2>0$

$= 7(2)^{5\lim_{x\to 1} x - 4}$ by constant multiple rule and rule 0.

$= 7(2)^{5\cdot 1-4}$ by rule 00.

$= 7(2)^{5-4} = 7\cdot 2^1 = 7\cdot 2 = 14$

b) $\lim_{x\to 0} \frac{\sin(x^2)}{2x^2} = \lim_{x\to 0} \frac{1}{2}\cdot\frac{\sin(x^2)}{x^2} = \frac{1}{2}\lim_{x\to 0}\frac{\sin(x^2)}{x^2}$ by cons multiply rule

Let $\theta = x^2$. Since $x^2 \to 0$ as $x\to 0$, we can replace $x\to 0$ with $\theta\to 0$ in the limit, as well.

$= \frac{1}{2}\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = \frac{1}{2}\cdot 1$ by $\frac{\sin\theta}{\theta}$ rule

$= \frac{1}{2}$

6) a) $f(x) = x^2 \sin(x)$

$(x^2 \sin x)' = (x^2)' \sin x + x^2 (\sin x)'$ by product rule

$= 2x^{2-1} \sin x + x^2 \cos x$ by power rule and sin rule.

$= 2x \sin x + x^2 \cos x$

b) $f(x) = \dfrac{x+4}{x^{10}}$

$\left(\dfrac{x+4}{x^{10}}\right)' = \dfrac{(x+4)' x^{10} - (x+4)(x^{10})'}{(x^{10})^2}$ by quotient rule

$= \dfrac{(x' + 4') x^{10} - (x+4) \cdot 10 x^{10-1}}{x^{20}}$ by sum rule and power rule

$= \dfrac{(1+0) x^{10} - (x+4) 10 x^9}{x^{20}}$ by x rule and constant rule

$= \dfrac{x^{10} - 10x^{10} - 40x^9}{x^{20}}$

$= \dfrac{-9x^{10} - 40x^9}{x^{20}} = \dfrac{-9}{x^{10}} - \dfrac{40}{x^{11}}$

Extra credit

a) I made a typo, it should have been

$$\lim_{x \to 2} \frac{x^2-4}{x-2}.$$

In this case, we are not justified in using the limit quotient rule since the limit of the denominator $=0$.

The problem as stated is incorrect because $\lim_{x \to 0} x^2-4 = -4$, $\lim_{x \to 0} x-2 = -2$, $\frac{-4}{-2} = 2$

b) When we substitute

$\theta = 2x + \pi/2$ here, θ does not go to zero as $x \to 0$, instead, $\theta \to \pi/2$.

So we can replace this with

$$\lim_{\theta \to \pi/2} \frac{\sin(\theta)}{\theta} = \frac{\lim_{\theta \to \pi/2} \sin\theta}{\lim_{\theta \to \pi/2} \theta} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi} \neq 1$$

So this is NOT the same as

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1.$$