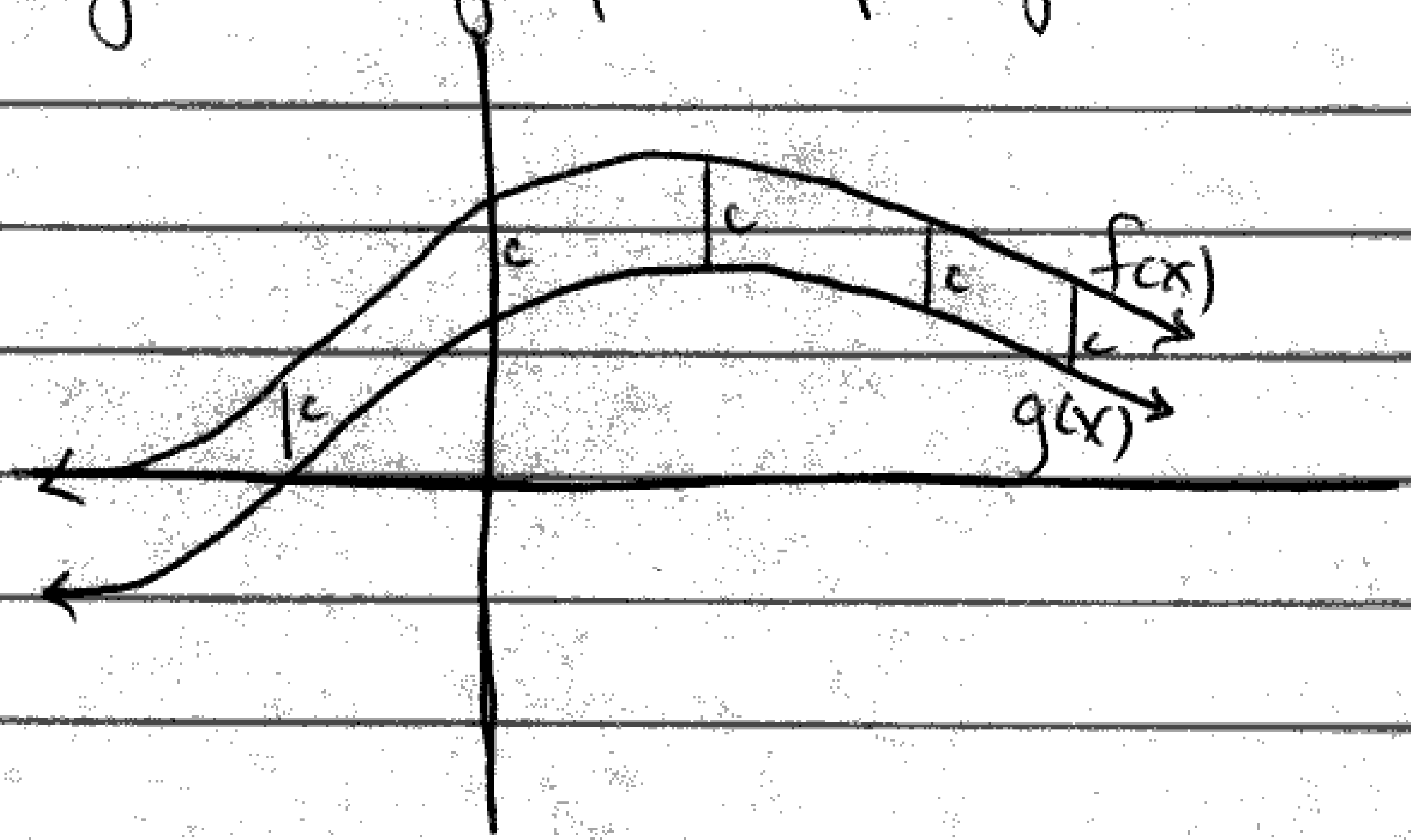
MIDTERM #2 SOLUTIONS

1 a) Let $(a,b) \subset D_g$. We say $f(x)$ is an antiderivative for $g(x)$ on (a,b) if $f'(x) = g(x)$ for all $x \in (a,b)$.

b) If $f'(x) = g'(x)$ then $f(x) = g(x) + c$ for some $c \in \mathbb{R}$ (by a corollary to the Mean Value Theorem). That is, the graph of $f(x)$ is ~~the~~ a vertical shift of the graph of $g(x)$:



c) If $h(t)$ = height of an object at time t, then
$h'(t)$ = velocity of the object at time t, and
$h''(t)$ = acceleration of the object at time t.
$= -3.7 \ m/s^2$

By b), any antiderivative of $h''(t) = -3.7$ is of the form $h'(t) = -3.7t + c$.
$h'(0) = -3.7 \cdot 0 + c = c$, So $c = V_0$, initial velocity.

Similarly, any antiderivative of $h'(t) = -3.7t + V_0$ is of the form $h(t) = \frac{-3.7}{2}t^2 + V_0 t + c$. (cont'd)

1 c) We have $h(0) = \frac{-3.7}{2} \cdot 0^2 + v_0 \cdot 0 + C = C$
So now $C = h_0$, initial height.

Thus we get $h(t) = \frac{-3.7}{2} t^2 + v_0 t + h_0$. We needed initial velocity and initial height in addition to acceleration.

d) On Mars:

$h(t) = \frac{-3.7}{2} t^2 + 0 \cdot t + 1.5$
$= -3.7/2\, t^2 + 1.5$
$0 = \frac{-3.7}{2} t^2 + 1.5$ when the hammer hits, so
$3.7 t_{mars}^2 = 3$
$t_{mars}^2 = \frac{3}{3.7}$
$t_{mars} \approx .9$ seconds

On Earth:

$h(t) = -\frac{9.8}{2} t^2 + 0 \cdot t + 1.5$
$= -\frac{9.8}{2} t^2 + 1.5$
this $= 0$ when
$\frac{9.8}{2} t^2 = 1.5$
$t_{earth}^2 = \frac{3}{9.8}$
$t_{earth} \approx .55$

So $t_{mars} - t_{earth} = .9 - .55 \approx .35$ seconds faster on Earth.

2) If you have questions about your essay, come see me. (Also look in the textbook)

3). $y = \#\text{carp}/\text{mile}$

x = pcb concentration in ppm. $\frac{dx}{dt} = 20$ ppm/year

$$y = \frac{1000}{1+x}$$

$$\frac{dy}{dt} = 1000 \frac{d}{dt}(1+x)^{-1}$$

$$= 1000\left(-(1+x)^{-2}\frac{dx}{dt}\right)$$

$$= \frac{-1000}{(1+x)^2}\cdot 20$$

$\frac{d}{dt}(1+x)^{-1}$:

$y = (1+x)^{-1} = u^{-1}$, $u = 1+x$

$\frac{dy}{du} = -u^{-2}$ $\qquad \frac{du}{dt} = \frac{dx}{dt}$

So $\frac{dy}{dt} = -u^{-2}\cdot\frac{dx}{dt} = \frac{-1}{(1+x)^2}\frac{dx}{dt}$

So when $x = 200$, $\frac{dy}{dt} = \frac{-20000}{(1+200)^2} \approx -.5$

So .5 fish/mile/year decrease, or 1 fish/2miles/year

4) a) We are optimizing (maximizing) A = annual yield/acre.
Other relevant quantities are
T = annual yield/tree
n = number of trees/acre
x = number of trees/acre over 55

4b) $A = Tn$

ie annual yield/acre = annual yield/tree · trees/acre

$n = x + 55$

$T = 352 - 5x$

c) $A = (352 - 5x)(x + 55)$

5a) If f is defined on (a,b) and attains a maximum at $c \in (a,b)$ and if $f'(c)$ is defined, then $f'(c) = 0$.

b) Critical values for a function f on a closed interval $[a,b]$ are c which satisfy one of:
- $f'(c) = 0$
- $f'(c)$ is undefined
- $c = a$ or $c = b$ (endpoints).

A corollary of the TIE is the critical point theorem which says that the maxima and minima of f must occur at critical values c.

In an optimization problem, we are looking for the conditions under which a certain quantity is maximized or minimized. Thus we need to find the critical values for that quantity, expressed in terms of some single variable.

Once we have the critical values, we plug them back into the function and compare results to determine which value is the optimal one for our problem. For instence, if we want to minimize cost, we find out the cost associated to each critical value. We know the smallest such cost is the actual minimum of the function.

c) The possible values for x are $x \in [0,10]$.

$$C(x) = 11{,}000\sqrt{25+x^2} + 10{,}000\,(10-x)$$

$$C'(x) = 11{,}000 \frac{d}{dx}\left(\sqrt{25+x^2}\right) + 10{,}000 \frac{d}{dx}(10-x)$$

Let $y = \sqrt{25+x^2} = \sqrt{u} = u^{\frac{1}{2}}$, $u = 25+x^2$

$$\frac{dy}{du} = \frac{1}{2}u^{-1/2} \qquad \frac{du}{dx} = 2x$$

$$\frac{dy}{dx} = \frac{2x}{2\sqrt{u}} = \frac{x}{\sqrt{u}} = \frac{x}{\sqrt{25+x^2}}$$

so

$$C'(x) = 11{,}000 \cdot \frac{x}{\sqrt{25+x^2}} - 10{,}000$$

This derivative is defined everywhere.

$C'(x) = 0$ where

$$0 = 11{,}000 \frac{x}{\sqrt{25+x^2}} - 10{,}000$$ simplify by $\div 1000$

$$0 = 11 \frac{x}{\sqrt{25+x^2}} - 10$$

$$10 = 11x / \sqrt{25+x^2}$$

$$\sqrt{25+x^2} = 11x/10 \implies 25+x^2 = \left(\tfrac{11}{10}\right)^2 x^2$$

$$25+x^2 = \tfrac{121}{100} x^2$$

$$25 = \tfrac{21}{100} x^2$$

$$\frac{2500}{21} = x^2$$

$$x = \sqrt{\frac{2500}{21}} \approx 10.91 \notin [0,10]$$

So the only critical points on this interval are the endpoints $x=0$, $x=10$

$$C(0) = 11{,}000\sqrt{25+0^2} + 10{,}000\,(10-0)$$
$$= 11{,}000 \cdot 5 + 100{,}000 = \$155{,}000$$

$$C(10) = 11{,}000\sqrt{25+10^2} + 10{,}000\,(10-10)$$
$$= 11{,}000\sqrt{125} \approx \$122{,}984$$

So \$122,984 is the lowest cost, and occurs when $x=10$.

(6) See graph on exam