Problem Set 7: Cake-cutting

Due at noon on 6/1

A much studied model in the fair division literature is *cake-cutting*. This, of course, is not prompted by a pressing need to resolve international disputes involving non-homogeneous cakes. But cake-cutting is a useful paradigm to study various procedures, and the "cake" can be seen as a stand-in for whatever you wish to divide fairly.

Harry and Sally just bought the Famous Non-homogeneous Banana Bread at the Fair Share bakery. Like all their cakes, this one came with a "you-can-split-your-cake-and-eat-it-too" manual. In this problem, we will follow their attempt to fairly share.

The "cake" will be represented by the unit interval, i.e. the interval $[0, 1]$ on the real number line. (Think of an elongated banana bread, with different goodies scattered in different parts of it.) The cake will be cut into two somewhere (this point is represented by a number $c \in [0, 1]$), and one part will go to Harry, the other to Sally. (See Figure 1 for a schematic rendering.) Given a "cut-off" number c, the utility function $u_H(c) = \sqrt{c}$ tells us what Harry's utility is of the left part, and the utility function $u_S(c) = c^2$ tells us what Sally's utility is of the left part. We will assume that given a cut-off number c, Harry's utility of the right part is $1 - u_H(c)$, and Sally's utility of the right part is $1 - u_S(c)$.

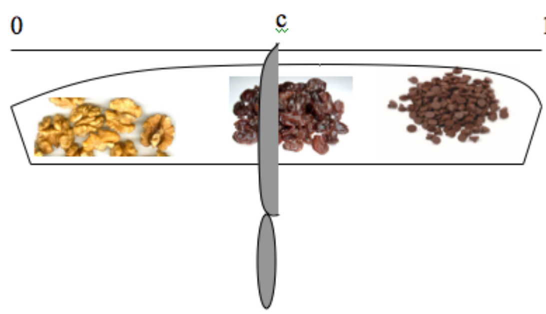

Figure 1: The generic banana bread

1 2 pts

One class of procedures for dividing a cake is the *moving knife* procedure. In its simplest version, the chef starts moving the knife from the left end slowly towards the right, at a constant speed. When either person says "Stop!", the knife stops and the cake is cut. The person who said "Stop!" gets the left part, and the other person gets the right part.

Harry and Sally do not know each other's utility functions. Before the knife starts moving, they each plan ahead. Harry thinks, "If Sally does not say "Stop!" by then, I will say it at the number c_H." Sally thinks, "If Harry does not say "Stop!" by then, I will say it at the number c_S." What number is c_H? What number is c_S? Justify your answer.

2 2 pts

Now Harry and Sally know each other's utility functions. Assuming that both are trying to maximize their utility, who will say "Stop!" first? At what number? Explain your answer. Is knowledge of the utility functions helpful to the two?

3 2 pts

Now we will explore what happens if Harry and Sally decide to use the Adjusted Winner procedure instead. For simplicity, we will modify the model slightly. We divide the cake into four equal parts, and we will specify, based on the above utility functions, the number of points that each person assigns to that part:

Part	Harry's value	Sally's value
$0 - \frac{1}{4}$	$\frac{100}{2}$	$\frac{100}{16}$
$\frac{1}{4} - \frac{1}{2}$	$\frac{100}{\sqrt{2}} - \frac{100}{2}$	$\frac{100}{4} - \frac{100}{16}$
$\frac{1}{2} - \frac{3}{4}$	$\frac{100\sqrt{3}}{2} - \frac{100}{\sqrt{2}}$	$\frac{100 \cdot 9}{16} - \frac{100}{4}$
$\frac{3}{4} - 1$	$100 - \frac{100\sqrt{3}}{2}$	$100 - \frac{100 \cdot 9}{16}$

Note that the total value for each person is 100, as required. For the rest of the problem, we will just use the values as given above, and disregard the previously given utility functions.

What allocation of the cake does the Adjusted Winner procedure result in? For full credit, you must show every step of the derivation of this allocation.

4 2 pts

Suppose that Harry knows Sally's values. Could Harry benefit from announcing different (false) values for the parts of the cake? If yes, give a set of values for Harry that will get him a bigger part of the cake under the Adjusted Winner procedure. For full credit, you must show every step of the derivation of the new allocation.