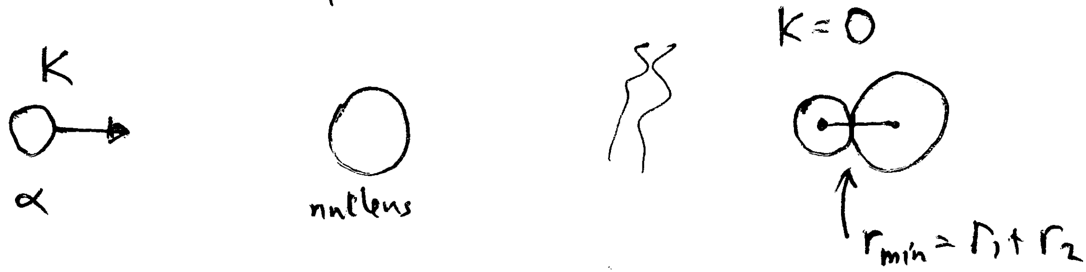
PHYSICS 160

Spring 2009

SOLUTIONS to Problem Set # 3

4-11 Nuclear radii of Al : 3.6 fm and Au : 7.0 fm.
Radii of alpha particles : 2.6 fm and protons : 1.3 fm.

(a) What energy α particles would be needed in head-on collisions for the nuclear surfaces to just touch?



Assuming an infinitely massive nucleus...

$$K_\alpha = \underbrace{\frac{q_1 q_2}{4\pi\epsilon_0 r_{min}}}_{\text{Potential energy at closest approach}} \qquad \text{where } q_1 = Z_1 e \text{ and } q_2 = Z_2 e$$

So.. $$K_\alpha = \frac{Z_1 Z_2 e^2}{4\pi\epsilon_0 (r_1 + r_2)}$$

If all quantities are in mks units, then K is in Joules. Divide by 1.6×10^{-19} J/eV $= e$ to get energy in eV. ~~Effectively,~~ just drop one factor of e.

For α : $Z_1 = 2$, Al. $Z_2 = 13$

- alphas approaching Aluminum $$K_\alpha(eV) = \frac{2\cdot13\cdot1.6\times10^{-19}}{(2.6+3.6)\times10^{-15}} \cdot \underbrace{8.99\times10^{9}}_{\frac{1}{4\pi\epsilon_0}}$$

$$K_\alpha = \frac{2\cdot13\cdot1.6\cdot8.99}{6.2}\times10^{15+9-19}$$

$$= 60\times10^5 = 6.0\times10^6 = \boxed{6.0\,\text{MeV}}$$

- alphas approaching Gold : $Z_2 = 79$

$$K_\alpha = \frac{2\cdot79\cdot1.6\times10^{-19}}{(2.6+7.0)\times10^{-15}}\, 8.99\times10^{9} = \frac{2\cdot79\cdot1.6\cdot8.99}{9.6}\times10^5 = 240\times10^5$$

$$= \boxed{24\,\text{MeV}}$$

(b) Now consider protons approaching ~~g~~ aluminum and gold nuclei.

- protons approaching Aluminum $Z_1 = 1$ $Z_2 = 13$ ...
  $r_1 = 1.3\,\text{fm}$ $r_2 = 3.6\,\text{fm}$

So... $$K_p(\text{eV}) = \frac{1 \cdot 13 \cdot 1.6\times10^{-19}}{(1.3+3.6)\times10^{-15}}\, 8.99\times10^{9}$$
$$= \frac{13 \cdot 1.6 \cdot 8.99}{4.9} \times 10^{5} = 38\times10^{5} = \boxed{3.8\text{ MeV}}$$

- protons approaching Gold
$$K_p'(\text{eV}) = \frac{1 \cdot 79 \cdot 1.6\times10^{-19} \cdot 8.99\times10^{9}}{(1.3+7.0)\times10^{-15}}$$
$$= \frac{79 \cdot 1.6 \cdot 8.99}{8.3} \times 10^{5} = 140 \times 10^{5}$$
$$= \boxed{14\text{ MeV}}$$

→ Higher incident particle energies are required for contact if the charge on either the incident or target particle is higher. The range of nuclear sizes is not large, so the distance of closest approach is a smaller effect that slightly reduces the necessary energy for larger incident and target particles.

$\boxed{4\text{-}22}$ What is the speed of the electron in the first three Bohr orbits of the Hydrogen atom?

The speed of the electron in the Bohr model is given in Eq. 4.31.

$$v_n = \frac{\hbar}{n m a_0} \quad \text{or..} \quad \frac{v_n}{c} = \frac{\hbar}{m c a_0} = \underbrace{\frac{\hbar c}{m c^2 a_0}}_{\text{fine structure constant } \alpha.} \cdot \frac{1}{n}$$

$$\alpha = \frac{\hbar c}{m c^2 \cdot a_0} = \frac{197.33\,\text{eV}\cdot\text{nm}}{511000\,\text{eV} \cdot 0.052918\,\text{nm}} = 0.00730 \cong \frac{1}{137}$$

So... for $n=1$ $\quad \frac{v_1}{c} = \alpha = 0.0073$

$n=2$ $\quad \frac{v_2}{c} = \frac{\alpha}{2} = 0.0036$

$n=3$ $\quad \frac{v_3}{c} = \frac{\alpha}{3} = 0.0024$

4-23 A hydrogen atom in an excited state ~~emits~~ absorbs a photon of wavelength 434 nm. What were the initial and final states of the hydrogen atom?

First determine the energy of the photon: $E_\gamma = \frac{hc}{\lambda} = \frac{1240\,eV\cdot nm}{434\,nm} = \underline{\underline{2.86\,eV}}$

Look in at Fig. 4.16 on p. 145, it is clear that the initial state must be $n_i = 2$. If n_i were equal to 1, all the required photon energies are greater than $13.6 - 3.40 = 10.2\,eV$. And if the initial state were $n_i = 3$ or greater, a 2.86 eV photon would ionize the atom. So...

Since
$$\underbrace{E_\gamma}_{2.86\,eV} = E_f - E_i = \underbrace{E_0}_{13.6\,eV}\left(\frac{1}{\underbrace{n_i^2}_{2}} - \frac{1}{n_f^2}\right)$$

Solve for n_f:
$$\frac{E_\gamma}{E_0} = \frac{1}{n_i^2} - \frac{1}{n_f^2} \quad \text{or} \quad \frac{1}{n_f^2} = \frac{1}{n_i^2} - \frac{E_\gamma}{E_0}$$

$$\frac{1}{n_f^2} = \frac{1}{4} - \frac{2.86}{13.6} = 0.0397 \quad \text{or} \quad n_f^2 = 25$$

$$\boxed{n_f = 5}$$

$$\boxed{n_i = 2}$$

$\boxed{4\text{-}24}$ A hydrogen atom in an excited state emits a photon of wavelength 95nm. What are the initial and final states of the hydrogen atom?

Find the energy of the photon: $E_\gamma = \frac{hc}{\lambda} = \frac{1240 eV\cdot nm}{95\ nm} = \underline{13.05 eV}$

The ~~photon~~ photon energy is close to the binding energy (13.6eV), so the transition must end on the ground state $n_f = 1$.

Photon energy equals the difference in electron energy between initial and final states...

$$E_\gamma = E_i - E_f = E_0\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right)$$

So... $\underbrace{\frac{1}{n_f^2}}_{1} - \frac{1}{n_i^2} = \frac{E_\gamma}{E_0} = \frac{13.05 eV}{13.61 eV} = 0.959$

solve for n_i...

$$\frac{1}{n_i^2} = 1 - 0.959 = 0.0408$$

$$n_i^2 = 24 \qquad \boxed{n_i \simeq 5} \quad \boxed{n_f = 1}$$

Verify.. $\Delta E = E_0\left(\frac{1}{1} - \frac{1}{25}\right) = 13.056 eV$ ...

$\boxed{4\text{-}25}$ What is the calculated binding energy of the electron in the ground state of

(a) deuterium. Deuterium is an isotope of hydrogen, so $Z = 1$.
The simplest Bohr model calculation assumes an infinite mass for the nucleus, so the fact that deuterium is heavier than hydrogen does not affect the simplest calculation of the binding energy.

The binding energy $E_0 = \dfrac{m_e Z^2 e^4}{2\hbar^2 (4\pi\epsilon_0)^2} = \underbrace{Z^2}_{1 \text{ for deuterium}} E_0^{\text{hydrogen}} = \boxed{13.6\,\text{eV}}$ ← 13.606eV to 5 sigfigs.

Modification of Eq. 4.26 to use with other 1 electron atoms/ions.

If we used the reduced mass correction to account for the finite mass of the nucleus (see section 4.5 on p. 148), we replace the electron mass m_e, with...

$$\mu = \frac{m_e}{1 + m_e/M_N}$$

For ~~deuterium~~ hydrogen $\dfrac{\mu}{m_e} = \dfrac{1}{1 + m_e/M_N} = 0.99946$ (see Example 4.8 on p. 148

$$E_0^{\text{hydrogen}} = 13.606 \times 0.99946 = \underline{13.599}\ \text{eV}$$

$$E_0^{\text{deuterium}} = 13.606 \times 0.99973 = \underbrace{\boxed{13.602\,\text{eV}}}_{\text{correction is in the } 5^{\text{th}} \text{ digit.}} = \boxed{13.6\,\text{eV}}$$

(b) He^+. Helium has a nuclear charge of $Z = 2$, so...
The reduced mass correction will be neglected. He^+ is a one electron ion.

$$E_0^{He^+} = 2^2 \cdot E_0^{\text{hydrogen}} = 4 \cdot 13.6\text{eV} = \boxed{54.4\,\text{eV}}$$

(c) Be^{+++}. Beryllium has nuclear charge $Z = 4$. So..

$$E_0^{Be^{3+}} = 4^2 E_0^{\text{hydrogen}} = 16 \cdot 13.6\text{eV} = \boxed{218\,\text{eV}}$$

4-29

A hydrogen atom exists in an excited state for typically 10^{-8} s. How many revolutions would an electron make in an $n=3$ state before decaying?

The velocity, from problem 4-22: $v_n = \frac{\alpha c}{n}$ where $\alpha = \frac{1}{137}$

The radius of the orbit is $r_n = a_0 n^2$ (Eq. 4-24)

So, the orbital period is... $T_n = \frac{2\pi r_n}{v_n} = \frac{2\pi a_0 n^3}{\alpha c}$

And the number of orbits in time τ is... $N = \frac{\alpha c \tau}{2\pi a_0 n^3}$

$\alpha = \frac{1}{137}$ $c = 3\times10^8$ m/s $a_0 = 0.53\times10^{-10}$ m, so...

$$N = \frac{3\times10^8 \text{ m/s} \cdot 10^{-8}\text{ s}}{137 \cdot 2\pi \cdot 0.53\times10^{-10}\text{ m} \cdot 3^3} = \frac{3}{137 \cdot 2\pi \cdot 0.53 \cdot 27} \cdot 10^{10} = 2.4\times10^{-4} \cdot 10^{10}$$

$\boxed{N \simeq 2\times10^6 \text{ orbits}}$

$\boxed{4-31}$ A muonic atom consists of a muon in place of an electron. For the muon in a hydrogen atom, what is

(a) the smallest radius? Mass of the muon $m_\mu = 106\ \mathrm{MeV}/c^2$

The Bohr radius $a_0 = \dfrac{4\pi\epsilon_0\hbar^2}{m_e e^2}$ becomes $a_0^\mu = \dfrac{4\pi\epsilon_0\hbar^2}{m_\mu e^2} = a_0 \cdot \dfrac{m_e}{m_\mu}$

$$a_0^\mu = \frac{0.511\ \mathrm{MeV}/c^2}{106\ \mathrm{MeV}/c^2} \cdot a_0 = \frac{a_0}{207} = 0.00255\ \cancel{\times 10^{-10}}\ \text{Å}$$

$$= \underline{2.55 \times 10^{-13}\ m}$$

The reduced mass correction is larger / more significant for a muonic atom..

$$\mu = \frac{m_\mu}{1 + m_\mu/M_N} \qquad M_N = 938\ \mathrm{MeV}/c^2 \text{ (see inside front cover)}$$

$$\frac{\mu}{m_\mu} = \frac{1}{1 + m_\mu/M_N} = \frac{1}{1 + 106/938} = 0.898$$

Using the reduced mass.. $a_0^\mu = \dfrac{a_0}{0.898 \cdot 207} = \underline{2.84 \times 10^{-13}\ m}$

(b) The binding energy of the muon in the ground state?

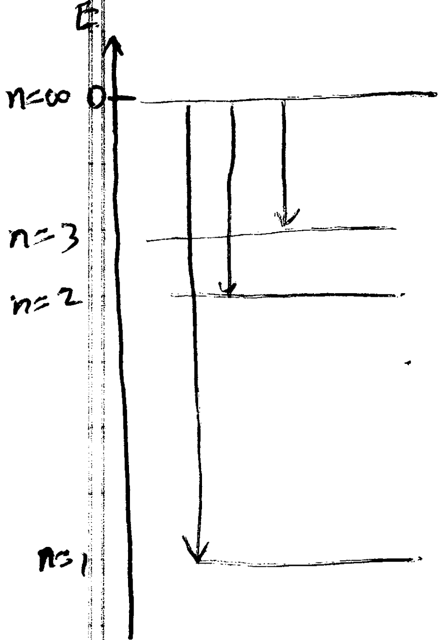
From Eq. 4.25 $\quad E_0 = \dfrac{e^2}{8\pi\epsilon_0 a_0}$

Using the result from part (a)...

$$E_0^\mu = E_0 \cdot \frac{a_0^e}{a_0^\mu} = 13.6\ eV \cdot \frac{0.529 \times 10^{-10}\ m}{2.84 \times 10^{-13}\ m}$$

$$= 2530\ eV = \boxed{2.53\ keV}$$

(c) Calculate the series limit of the wavelength for the first three series....



The series limit corresponds to the most energetic photon emitted for transitions that terminate on... n = 1, 2, 3

n=1 series (Lyman series in Hydrogen)...

$E_\gamma = E_0^M = 2.53\ \text{keV}$

$$\lambda = \frac{hc}{E} = \frac{1240\ \text{eV}\cdot\text{nm}}{2530\ \text{eV}} = \boxed{0.490\ \text{nm}}$$

n=2 series (Balmer Series in Hydrogen):

$$E_\gamma = \frac{E_0^\gamma}{2^2} = \frac{2530\text{eV}}{4} = 633\text{eV}$$

$$\lambda = \frac{hc}{E} = \frac{1240\text{eV nm}}{633\text{eV}} = \boxed{1.96\ \text{nm}}$$

n=3 series (Paschen series in Hydrogen):

$$E_\gamma = \frac{E_0^\gamma}{3^2} = \frac{2530\ \text{eV}}{9} = 281\text{eV}$$

$$\lambda = \frac{hc}{E} = \frac{1240\text{eV}\cdot\text{nm}}{281\ \text{eV}} = \boxed{4.43\ \text{nm}}$$

$\boxed{4\text{-}37}$ Singly-ionized helium is a one-electron ion with $Z = 2$.

In the derivation of the Bohr model, a nuclear charge of $+Ze$ rather than $+e$ alters the following equations...

Eq 4.21 $$E = \frac{-Ze^2}{8\pi\epsilon_0 r}$$

Eq. 4.23 $$v^2 = \frac{Ze^2}{4\pi\epsilon_0 m r} = \frac{n^2\hbar^2}{m^2 r^2}$$

Eq. 4.24 $$r_n = \frac{4\pi\epsilon_0 n^2\hbar^2}{mZe^2} = \frac{n^2 a_0}{Z}$$

and Eq. 4.25 $$E_n = -\frac{Ze^2}{8\pi\epsilon_0 \left(\frac{a_0}{Z}\right) n^2} = \frac{-Z^2 E_0}{n^2}$$

The wavelengths of the emission spectrum are....

Eq. 4.28 $$\frac{1}{\lambda} = \underbrace{\frac{-Z^2 E_0}{hc}}\left(\frac{1}{n_\ell^2} - \frac{1}{n_u^2}\right)$$

effective Rydberg constant must be altered for nuclear charge $+Ze$...,

where $$R_{He^+} = \frac{4E_0}{hc} = 4R_\infty$$

using R_∞ (no reduced mass correction)..

$$R_{He^+_\infty} = 4 \cdot 1.09737 \times 10^7\ \text{m}^{-1} = \boxed{4.38948 \times 10^7\ \text{m}^{-1}}$$

reduced mass correction.. $$R_{He^+} = \frac{\mu}{m_e} R_{He^+_\infty} = \frac{1}{1 + \frac{m_e}{M_{He^+}}} R_{He^+_\infty} = 0.99986 \cdot R_\infty = \underline{4.38888 \times 10^7\ \text{m}^{-1}}$$

$\boxed{5-11}$ Determine the deBroglie wavelength of a particle of mass m and kinetic energy K.

(a) <u>Relativistic</u>:

The deBroglie wavelength is $\lambda = \frac{h}{p}$ independent of whether the particle is relativistic or not. However, we need to relate the ~~p~~ momentum to the kinetic energy.

$$K = E - mc^2 \quad \text{and} \quad E = \sqrt{p^2c^2 + m^2c^4}$$

So.. $K = \sqrt{p^2c^2 + m^2c^4} - mc^2$

or.. $(K + mc^2)^2 = p^2c^2 + m^2c^4$

or $pc = \sqrt{(K + mc^2)^2 - m^2c^4}$

$$\boxed{\lambda = \frac{hc}{pc} = \frac{hc}{\sqrt{(K + mc^2)^2 - m^2c^4}}}$$

(b) Nonrelativistically $K = \frac{p^2}{2m}$ So.. $p = \sqrt{2mK}$

and $\boxed{\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mK}}}$

For a little more fun, ... show that the relativistic equation reduces to the non-relativistic one when $K << mc^2$

Note that we can rewrite the equation for λ in part (a) ...

$$\lambda = \frac{hc}{mc^2\left[\underbrace{\left(\frac{K}{mc^2}+1\right)^2}_{\text{multiply out...}} - 1\right]^{\frac{1}{2}}}$$

$$\lambda = \frac{hc}{mc^2\left[\left(\left(\frac{K}{mc^2}\right)^2 + 2\frac{K}{mc^2} + 1 - 1\right)\right]^{\frac{1}{2}}}$$

these ones cancel

If $K << mc^2$ then this term is negligible compare to the other one.

$$\lambda \cong \frac{h\cancel{c}}{m\cancel{c^2}\sqrt{2\frac{K}{mc^2}}} = \frac{h}{\sqrt{m^2c^2 \cdot 2\frac{K}{mc^2}}} = \underline{\underline{\frac{h}{\sqrt{2mK}}}}$$

check ✓

$\boxed{5-13}$ Find the kinetic energy of particles with a deBroglie wavelength of 0.15nm

(a) photons ... All of a photon's energy is kinetic $(m=0)$, so...

$$K = E = \frac{hc}{\lambda} = \frac{1240\,\text{eV}\cdot\text{nm}}{0.15\,\text{nm}} = 8267\,\text{eV} = \underline{\underline{8.3\,\text{keV}}}$$

(b) electrons... Assume non-relativistic ~~pat~~ relation between kinetic energy and momentum... then verify.

See 5-11.
$$\lambda = \frac{h}{\sqrt{2mK}}$$

Solve for K...
$$\sqrt{2mK} = \frac{h}{\lambda}$$

$$2mK = \frac{h^2}{\lambda^2} \quad \text{or..} \quad K = \frac{h^2}{2m\lambda^2} = \frac{(hc)^2}{2(mc^2)\lambda^2}$$

$$K = \frac{(1240\,\text{eV}\cdot\text{nm})^2}{2(511{,}000\,\text{eV})(0.15\,\text{nm})^2} = \boxed{67\,\text{eV}}$$

This is much less than the rest mass energy and validates the use of non-relativistic equations.

(c) neutrons: Use the relation from part (b): $K = \dfrac{(hc)^2}{2(mc^2)\lambda^2}$

but $mc^2 = 940\times10^6\,\text{eV}$ (from front cover of the textbook).

So...
$$K = \frac{(1240\,\text{eV}\cdot\text{nm})^2}{2(940\times10^6\,\text{eV})\cdot(0.15\,\text{nm})^2} = \boxed{0.036\,\text{eV}}$$

(d) alpha particle : Again, use $K = \frac{(hc)^2}{2(mc^2)(\lambda^2)}$

$mc^2 = 3727 \times 10^6$ eV (front cover)

$$K = \frac{(1240 eV \cdot nm)^2}{2(3727 \times 10^6 eV)(.15 nm)^2} = \boxed{0.0092 eV}$$

5-15 An electron initially at rest is accelerated across a potential difference of 3.00 kV. What are its wavelength, momentum, kinetic energy, and total energy?

- Kinetic energy is $\boxed{3.00 \text{ keV}}$ This follows from
  - conservation of energy, and
  - definition of 1 eV.

- ~~Wavelength ... from~~
- Momentum... $p = \sqrt{2mK}$ (non-relativistic since $K \ll mc^2$)

or... $pc = \sqrt{2(mc^2)K}$

$$= \sqrt{2(511\,000)(3000)} = \sqrt{2(511)(3)} \text{ keV}$$

$$\boxed{p = 55 \text{ keV}/c}$$

- Wavelength $\lambda = \frac{h}{p} = \frac{hc}{pc} = \frac{1240 \text{ eV}\cdot\text{nm}}{55000 \text{ eV}} =$

$$\boxed{\lambda = 0.023 \text{ nm}}$$

- Total Energy $E = K + mc^2 = 3.00 \text{ keV} + 511 \text{ keV}$

$$\boxed{E = 514 \text{ keV}}$$

5-36 Proton confined in a uranium ~~nucleus~~ nucleus: $R = 8\times10^{-15}$ m.

If we approximate the ~~box~~ nucleus as a 1D box with

$\Delta x = \frac{D}{2} = R = 8\times10^{-15}$ m then, according to the uncertainty principle..

$$\Delta p \geq \frac{\hbar}{2\Delta x} = \frac{\hbar c}{2c\Delta x} = \frac{\hbar c}{c\,D}$$

diameter

The minimum kinetic energy of the proton is...

$$K_{min} = \frac{\Delta p^2}{2m} = \frac{(\hbar c)^2}{2mc^2 D^2}$$

[Recall the argument presented in class that $\Delta p^2 = \langle p^2 \rangle = 2mK$]

$$K_{min} = \frac{(197\text{ eV}\cdot\text{nm})^2}{2(938\times10^6\text{ eV})(8\times10^{-6}\text{ nm})^2}$$

$$\boxed{K_{min} = 323\text{ keV}}$$

$\boxed{5\text{-}40}$ The lifetime of the proton, in some theories, is about 10^{36} yrs. What would such a prediction say about the energy of the proton?

Uncertainty principle $\qquad \Delta E \Delta t \geq \frac{\hbar}{2}$

If $\Delta t = 10^{36}\,\text{yrs} \cdot 365\,\frac{d}{yr} \cdot 24\,\text{hr}/d \cdot 3600\,\text{s}/\text{hr}$

$$= 3. \times 10^{43}\,s$$

Then $\quad \Delta E \geq \frac{\hbar}{2\Delta t} = \frac{\hbar c}{2c\Delta t}$

$$= \frac{197\,eV\cdot nm}{2\cdot 3\times10^{17}\,\frac{nm}{s}\cdot 3\times10^{43}\,s} = \boxed{10^{-59}\,eV}$$

This is the minimum uncertainty in the proton's total energy --

$$\frac{\Delta E}{E} \geq \frac{10^{-59}\,eV}{938\times10^{6}} \simeq \underline{10^{-68}}$$ relative minimum uncertainty

The proton's mass/energy can, in principle, be known to exceedingly high precision.

$\boxed{5-41}$ What is the bandwidth $\Delta\omega$ of an amplifier for radar if it amplifies pulses of width $2.4\,\mu s$?

Classical uncertainty principle:

$$\Delta\omega\,\Delta t \geq \frac{1}{2}$$

$$\Delta\omega \geq \frac{1}{2\Delta t} = \frac{1}{2\cdot 2.4\times10^{-6}\,s} = \underline{2.08\times10^{5}\ s^{-1}}$$

$$\Delta f = \frac{\Delta\omega}{2\pi} = \underline{\underline{33\ kHz}}$$