Problem Set #2: Due in class on Wed. 4/15

Problems from Chapter 2 of Thornton & Rex: 67, 70, 76, 84, 87

Problems from Chapter 3 of Thornton & Rex: 18, 19, 20, 34, 35, 39, 47, 52

Additional Problems:

A. Reproduce the spacetime diagram for the outbound leg of the Twin Paradox handout. Use it to visualize the length contraction effect. To do so complete the following steps:
   a. Use the Lorentz transformation equations to show that the x' axis has slope of v/c.
   b. Show that the distance scale is marked as shown in the handout.
   c. Consider a 10 l.y. long meter stick (proper length) that is stationary in Tom's frame, observed from Emily's frame. If Emily measures the length of the meter stick by determining the position of both ends of the meter stick *simultaneously* at t=0, what length does she obtain? How is it displayed on the spacetime diagram?

B. Fill in the missing steps in the derivation of the relativistic kinetic energy on p. 64, between equations 2.57 and 2.58.

C. Relativistic freefall: Determine the velocity versus time for a particle, initially at rest, subjected to a constant force. Recall that Newton's second law can be written $F = dp/dt$ as long as you use the relativistic momentum. First find the relativistic momentum as a function of time. Then solve for the velocity versus time and compare it to the "old" result where momentum has the form $p = mu$.

$\boxed{2-67}$ Calculate the momentum, kinetic energy, and total energy of an electron traveling at a speed of (a) 0.01c, (b) 0.1c, and (c) 0.9c.

Relativistic momentum $p = \dfrac{mu}{\sqrt{1-u^2/c^2}}$

Kinetic energy $K = mc^2\left[\dfrac{1}{\sqrt{1-u^2/c^2}} - 1\right]$

Total energy $E = \dfrac{mc^2}{\sqrt{1-u^2/c^2}} = K + mc^2$

It is helpful to know the rest mass energy of the electron: $m_e c^2 = 511\ \text{keV} = 8.19\times10^{-14}\ \text{J}$

Electron volts are used more commonly than Joules.

Momentum can be expressed in units of eV/c (or keV/c, or MeV/c), as follows...

$$p = \frac{mu}{\sqrt{1-u^2/c^2}} = \frac{mc^2\cdot\left(\frac{u}{c}\right)}{c\sqrt{1-(u/c)^2}} = \underbrace{\frac{mc^2\,(\text{eV})}{c}}_{\text{units of eV/c}}\cdot\underbrace{\frac{u/c}{\sqrt{1-u^2/c^2}}}_{\text{dimensionless}}$$

For electrons $p = 511\ \text{keV}/c\cdot\left[\dfrac{u/c}{\sqrt{1-u^2/c^2}}\right]$ or $= \underbrace{\dfrac{8.19\times10^{-14}\ \text{J}}{3\times10^8\ \text{m/s}}}_{2.73\times10^{-22}\ \frac{\text{kg m}}{\text{s}}}\cdot\dfrac{u/c}{\sqrt{1-u^2/c^2}}$

(a) If $u/c = 0.01$, then $\dfrac{1}{\sqrt{1-u^2/c^2}} \simeq 1 + \dfrac{u^2}{2c^2}$ first two terms in power series for $u/c \ll 1$

$$\simeq 1 + \frac{1}{2}\left(\frac{1}{100}\right)^2 = 1 + \underbrace{\frac{1}{2}\cdot10^{-4}}_{\text{negligible!}} \simeq \underline{1.00}$$

Momentum

So... $p = \dfrac{mc^2}{c}\cdot u/c = 511\ \text{keV}/c\cdot 0.01 = \boxed{5.11\ \text{keV}/c}$

$$= 2.73\times10^{-22}\ \frac{\text{kg m}}{\text{s}}\cdot 0.01 = \boxed{2.73\times10^{-24}\ \text{kg m/s}}$$

kinetic energy

$$K \cong mc^2\left[1 + \frac{1}{2}\left(\frac{u}{c}\right)^2 - 1\right] \cong \frac{1}{2}\, mc^2 \cdot \left(\frac{u}{c}\right)^2$$

$$= \frac{1}{2} \cdot 511\,\text{keV} \cdot \left(\frac{1}{100}\right)^2 = 2.56\times10^{-2}\,\text{keV} = \boxed{25.6\,\text{eV}}$$

$$= \frac{1}{2} \cdot 8.19\times10^{-14}\,\text{J} \cdot \left(\frac{1}{100}\right)^2 = \boxed{4.10\times10^{-18}\,\text{J}}$$

Total energy : $E = \dfrac{mc^2}{\sqrt{1-u^2/c^2}} \simeq mc^2\left[1 + \frac{1}{2}\left(\frac{u^2}{c^2}\right)\right] \simeq mc^2$

negligible...

$$\boxed{E = 511\,\text{keV} = 8.19\times10^{-14}\,\text{J}}$$

Kinetic energy is much less than rest mass energy, so $E \simeq mc^2$.

(b) If $u/c = 0.1c$, then $\dfrac{1}{\sqrt{1-u^2/c^2}} = \dfrac{1}{\sqrt{1-\left(\frac{1}{10}\right)^2}} = \dfrac{1}{\sqrt{0.99}} = 1.0050$

momentum :

$$p = 511\,\text{keV}/c \cdot \frac{1}{10} \cdot 1.0050 = \boxed{51.4\,\text{keV}/c}$$

$$= 2.73\times10^{-22}\,\frac{\text{kg m}}{\text{s}} \cdot \frac{1}{10} \cdot 1.0050 = \boxed{2.74\times10^{-23}\,\frac{\text{kg m}}{\text{s}}}$$

kinetic energy : $K = mc^2\left[\dfrac{1}{\sqrt{1-u^2/c^2}} - 1\right] = 511\,\text{keV} \cdot \left[1.0050 - 1.0\right]$

$$= 0.0050 \cdot 511\,\text{keV} = \boxed{2.56\,\text{keV}}$$

$$= 0.0050 \cdot 8.19\times10^{-14}\,\text{J} = \boxed{4.10\times10^{-16}\,\text{J}}$$

Total energy : $E = \dfrac{mc^2}{\sqrt{1-u^2/c^2}} = 511\,\text{keV} \cdot 1.0050 = \boxed{514\,\text{keV}}$

$$= 8.19\times10^{-14}\,\text{J} \cdot 1.0050 = \boxed{8.23\times10^{-14}\,\text{J}}$$

(c) If $u/c = 0.9$ then $\frac{1}{\sqrt{1-u^2/c^2}} = \frac{1}{\sqrt{1-\left(\frac{9}{10}\right)^2}} = \frac{1}{\sqrt{\frac{19}{100}}} = \frac{10}{\sqrt{19}} = \underline{\underline{2.29}}$

So... momentum: $p = 511\ \mathrm{keV}/c \cdot \left(\frac{9}{10}\right) \cdot 2.29 = 1053\ \mathrm{keV}/c = \boxed{1.05\ \mathrm{MeV}/c}$

$= 2.73 \times 10^{-22}\ \mathrm{kg\frac{m}{s}} \cdot \left(\frac{9}{10}\right) 2.29 = \boxed{5.63 \times 10^{-22}\ \mathrm{kg\, m/s}}$

kinetic energy: $K = mc^2\left[\frac{1}{\sqrt{1-u^2/c^2}} - 1\right] = 511\ \mathrm{keV} \cdot \left[2.29 - 1.0\right] = 511\ \mathrm{keV} \cdot 1.29$

$= \boxed{659\ \mathrm{keV}}$

$= 8.19 \times 10^{-14}\ \mathrm{J} \cdot 1.29 = \boxed{1.057 \times 10^{-13}\ \mathrm{J}}$

total energy: $E = mc^2\left[\frac{1}{\sqrt{1-u^2/c^2}}\right] = 511\ \mathrm{keV} \cdot 2.29 = 1170\ \mathrm{keV}$

$= \boxed{1.170\ \mathrm{MeV}}$

$= 8.19 \times 10^{-14}\ \mathrm{J} \cdot 2.29 = \boxed{1.88 \times 10^{-13}\ \mathrm{J}}$

$\boxed{2-70}$ What is the speed of a proton when its kinetic energy is equal to twice its rest energy?

IF $K = 2mc^2$, then $E = K + mc^2 = 3mc^2$

Since $E = \frac{mc^2}{\sqrt{1-u^2/c^2}}$, it must be the case that

$$\frac{1}{\sqrt{1-u^2/c^2}} = 3$$ independent of the absolute value of the rest mass energy.

Solve for u/c: $\frac{1}{1-u^2/c^2} = 9$ invert both sides..

$$1 - u^2/c^2 = \frac{1}{9}$$

$$\frac{u^2}{c^2} = 1 - \frac{1}{9} = \frac{8}{9}$$ or.... $$\frac{u}{c} = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3} =$$

$$\boxed{u/c = 0.943}$$

$$= 2.83 \times 10^8 \text{ m/s}$$

2-76 Calculate the energy needed to accelerate a spaceship of mass 10^4 kg to a speed of $0.3c$. Compare with the annual energy usage on earth of 10^{21} J.

The energy needed to accelerate the spaceship to $0.3c$ from rest is equal to its final kinetic energy ...by the work energy theorem:

$$K = mc^2\left[\frac{1}{\sqrt{1-u^2/c^2}} - 1\right] = (10^4\,\text{kg})(3\times10^8\,\text{m/s})^2\cdot\left[\frac{1}{\sqrt{1-\left(\frac{3}{10}\right)^2}} - 1\right]$$

$$= 10^4\,\text{kg}\cdot 9\times10^{16}\,\text{m}^2/\text{s}^2\cdot\left[\frac{1}{\sqrt{\frac{91}{100}}} - 1\right]$$

$$= 9\times10^{20}\,\text{J}\cdot\left[\frac{10}{\sqrt{91}} - 1\right]$$

$$= 9\times10^{20}\,\text{J}\cdot[0.048] = \boxed{4.3\times10^{19}\,\text{J}}$$

This represents ... $\frac{10^{21}}{4.3\times10^{19}} = \frac{100}{4.3} \simeq$ 23 years of Earth's energy usage.

$\boxed{2\text{-}84}$ Fusion reaction $^2H + ^2H \rightarrow n + ^3He$

(a) Calculate the ~~mass~~/energy imbalance to determine how much ~~energy~~ mass is converted to energy in this reaction:

Masses

^2H : 2.014102 u

n : 1.008665 u

^3He : 3.016029 u

So the initial mass is $M_i = 2(2.014102)\,u = 4.028204\,u$

The final mass is $M_f = 1.008665\,u + 3.016029\,u = 4.024694\,u$

The difference is... $\Delta M = 0.00351\,u \times 931.49 \frac{MeV}{c^2 \cdot u} = \boxed{3.27\,MeV/c^2}$

(b) What percentage of the initial mass is converted to kinetic energy?

$$\frac{\Delta M}{M_i} = \frac{3.51 \times 10^{-3}}{4.028} = 8.7 \times 10^{-4} \Rightarrow 8.7 \times 10^{-2}\,\%$$

$$= \boxed{0.087\%}$$

Less than 1%.

$\boxed{2-87}$ An Ω^- particle has rest mass energy of 1672 MeV and a mean lifetime (measured at rest) of 8.2×10^{-11} s. It is created and decays in a particle track detector and leaves a 24 mm long track. What is the total energy of the Ω^- particle?

Since $E = \frac{mc^2}{\sqrt{1-u^2/c^2}}$ and, we are given $mc^2 = 1672$ MeV, we need to determine the Ω^- particle's velocity.

$$u = \frac{L}{\Delta t}$$
$\leftarrow$ length of track
$\leftarrow$ time required to make the track before decaying... measured in the lab frame.

Due to time dilation $\Delta t = \frac{\Delta t_0}{\sqrt{1-u^2/c^2}}$ where Δt_0 is the mean lifetime...

So... $u = \frac{L\sqrt{1-u^2/c^2}}{\Delta t_0}$ or... $u^2 = \frac{L^2}{\Delta t_0^2}\left(1-u^2/c^2\right)$

Solve for u... $$\left(\frac{u^2}{c^2}\right) = \frac{L^2}{c^2\Delta t_0^2}\left(1-\frac{u^2}{c^2}\right)$$

or. $$\left(\frac{u^2}{c^2}\right)\left[1+\frac{L^2}{c^2\Delta t_0^2}\right] = \frac{L^2}{c^2\Delta t_0^2}$$

$$\frac{u}{c} = \frac{L/c\Delta t_0}{\sqrt{1+(L/c\Delta t_0)^2}} \overset{or}{=} \frac{1}{\sqrt{1+\left(\frac{c\Delta t_0}{L}\right)^2}}$$

$$\frac{c\Delta t_0}{L} = \frac{3\times10^8\,\text{m/s}\cdot 8.2\times10^{-11}\,\text{s}}{2.4\times10^{-2}\,\text{m}} = \frac{3\cdot 8.2}{2.4}\cdot 10^{-1} = \underline{1.025}$$

$$\frac{u}{c} = \frac{1}{\sqrt{2.025}} = \underline{\underline{0.703}}$$

So... $$E = 1672\text{ MeV}\cdot\left[\frac{1}{\sqrt{1-(.703)^2}}\right] = 1672\text{ MeV}\cdot 1.41 = \boxed{2350\text{ MeV}}$$

3-18 Calculate the temperature of a blackbody if the spectral distribution peaks at (a) $\lambda_{max} = 1.00 \times 10^{-14}$ m, (b) 1.00 nm, (c) 670 nm, (d) 1.00 m, and (e) 204 m

Use Wien's Law : $\lambda_{max} T = 2.898 \times 10^{-3}$ m·K

so... $T = \dfrac{2.898 \times 10^{-3} \text{ m·K}}{\lambda_{max} \text{ (m)}}$

	Band	λ_{max}	T (K)
(a)	γ-ray	10^{-14} m	2.9×10^{11} K
(b)	X-ray	10^{-9} m	2.9×10^{6} K
(c)	red	6.7×10^{-7} m	4300 K
(d)	TV rf	1 m	2.9 mK = 2.9×10^{-3} K
(e)	AM	204 m	1.42×10^{-5} K

$\boxed{3-19}$ A blackbody's temperature is increased from 900K to 1900K. By what factor does the total power radiated per unit area increase?

Use the Stefan-Boltzmann Law $\quad R = \sigma T^4$

Use ratios ... $\quad \dfrac{R_2}{R_1} = \left(\dfrac{T_2}{T_1}\right)^4$

So, if $\dfrac{T_2}{T_1} = \dfrac{1900}{900} = 2.11$ then...

$$\frac{R_2}{R_1} = 2.11^4 = \underline{\underline{20}}$$

The radiated power per unit area increases by a factor of 20

3-20 (a) At what wavelength will the human body radiate the maximum radiation?

Body Temperature = 98.6 °F

$$T(°C) = \frac{5}{9} \times (T(°F) - 32°)$$

$$= \frac{5}{9}(98.6 - 32)$$

$$T(°C) = 37\,°C$$

$$T(K) = T(°C) + 273$$

$$\boxed{T(K) = 310\,K}$$

Use Wien's Law: $\lambda_{max} T = 2.898 \times 10^{-3}\ m \cdot K$

So... $\lambda_{max} = \frac{2.898 \times 10^{-3}}{310}\ m = 9.35 \times 10^{-6}\ m \cong \boxed{9\,\mu m}$

Infrared

(b) Estimate the total radiated power ... assuming a cylindrical human of height 165 cm and radius 13 cm

Use Stefan-Boltzmann Law $R = \sigma T^4$ ← this is power per unit area...

$$R = 5.67 \times 10^{-8} \frac{W}{m^2 K^4} \cdot (310K)^4 = 524 \frac{W}{m^2}$$

Multiply by the surface area of the cylinder to get total power: $A = 2\pi r \ell + 2\pi r^2$

side ↑ ↑ ends

$$A = 2\pi(r\ell + r^2) = 2\pi(0.13m \cdot 1.65m + 0.13m^2)$$

$$= 1.45\ m^2$$

So... $\boxed{P_{rad} = 760\ W}$ ← This assumes the person is not absorbing radiation from her surroundings. If surroundings are cold, the body's net radiated power is greater than if the surroundings are warmer.

2-34 What is the threshold frequency for the photoelectric effect on lithium ($\phi = 2.93\,eV$)?

At the threshold frequency, the photon energy is just enough to liberate the electron from the metal...

$$hf_0 = \phi$$

So..

$$f_0 = \frac{\phi}{h} = \frac{2.93\,eV}{4.14\times10^{-15}\,eV\cdot s} = \boxed{7.08\times10^{14}\,Hz}$$

This corresponds to a wavelength of... $\lambda_0 = \frac{c}{f_0} = \underline{424\,nm}$ violet

What is the stopping potential if the wavelength of the incident light is 400 nm?

$$eV_0 = hf - \phi \quad \text{or...} \quad eV_0 = \frac{hc}{\lambda} - \phi$$

Photon energy $\frac{hc}{\lambda} = \frac{1240\,eV\cdot nm}{400\,nm} = 3.10\,eV$

So... $eV_0 = 3.10\,eV - 2.93\,eV = \underline{0.17\,eV}$.

or $\boxed{V_0 = 0.17\,V}$

$\boxed{3\text{-}35}$ What is the maximum wavelength that can produce photoelectrons from silver ($\phi = 4.64\,eV$)?

Max wavelength corresponds the threshold frequency ...or when the photon energy equals the work function...

$$\frac{hc}{\lambda_{max}} = \phi \quad \text{or...} \quad \lambda_{max} = \frac{hc}{\phi}$$

$$\text{So...} \quad \lambda_{max} = \frac{1240\,eV\cdot nm}{4.64\,eV} = \boxed{267\,nm} \quad UV$$

What will be the maximum kinetic energy of the photoelectrons if the wavelength is halved?

$$\lambda = 133\,nm$$

$$\text{Then} \quad K_{max} = \underbrace{\frac{hc}{\lambda}}_{E_\gamma} - \phi = \frac{1240\,eV\cdot nm}{133\,nm} - 4.64\,eV$$

$$= (9.32 - 4.64)\,eV$$

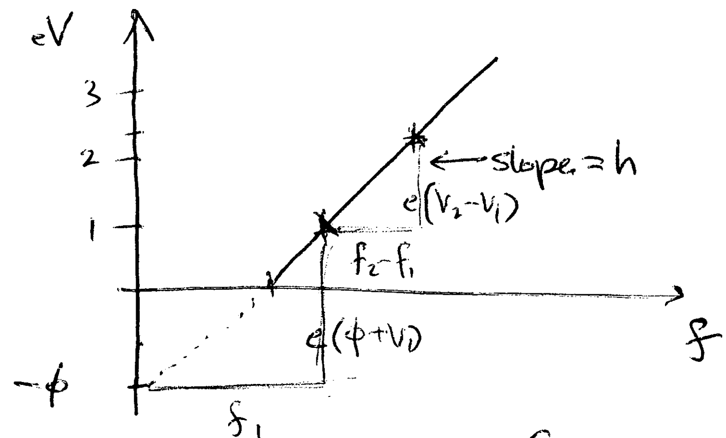
$$\boxed{K_{max} = 4.68\,eV} = 7.49\times10^{-19}\,J$$

3-39 Photoelectric effect experiment data :

	Wavelength (nm)	Stopping Potential (V)
①	260	1.00
②	207	2.30

Find Planck's Constant and the Stopping Potential

Einstein says... $eV_0 = \frac{hc}{\lambda} - \phi$



Calculate the frequencies.. $f_1 = \frac{c}{\lambda_1} = 1.15 \times 10^{15}\,Hz$

$f_2 = \frac{c}{\lambda_2} = 1.45 \times 10^{15}\,Hz$

Planck's constant is the slope...

$$h = \frac{e(V_2 - V_1)}{f_2 - f_1} = \frac{1.30\,eV}{0.30 \times 10^{15}\,Hz} = 4.33 \times 10^{-15}\,eV \cdot s$$

$$* 1.60 \times 10^{-19} \frac{J}{eV} = \boxed{6.93 \times 10^{-34}\,J \cdot s}$$

Work function :

$$h = \frac{e\phi + eV_1}{f_1} \quad \text{solve for } \phi ...$$

$$\phi = hf_1 - V_1 = 4.33 \times 10^{-15}\,eV \cdot s \cdot 1.15 \times 10^{15}\,Hz - 1.00\,eV$$

$$= 4.98 - 1.00 = \boxed{3.98\,eV}$$

$$= 6.37 \times 10^{-19}\,J$$

3-47

A photon having 40 keV scatters from a free electron at rest. What is the maximum energy that the electron can obtain?

The electron receives the maximum energy when the Compton scattering angle is 180° ... In that case the photon suffers a wavelength shift of

$$\Delta\lambda_{max} = 2\lambda_c = 0.00485 \text{ nm}$$

The initial wavelength of the 40keV photon is $\lambda_1 = \frac{hc}{hf_1} = \frac{hc}{E_1} = \frac{1240 \text{ eV}\cdot\text{nm}}{40{,}000 \text{ eV}}$

$$\lambda_1 = 0.0310 \text{ nm}$$

So the wavelength of the scattered photon is .. $\lambda_2 = \lambda_1 + \Delta\lambda_{max}$

$$= \underline{0.0359 \text{ nm}}$$

And final ~~wavelength~~ energy of the photon is ..

$$E_2 = \frac{hc}{\lambda_2} = \frac{1240 \text{ eV}\cdot\text{nm}}{0.0359 \text{ nm}} = \underline{34.5 \text{ keV}}$$

The electron therefore must receive energy... $E_{electron} = E_1 - E_2 = \boxed{\underline{5.5 \text{ keV}}}$

$$= \underline{8.8\times10^{-16} \text{ J}}$$

$\boxed{3\text{-}52}$

A gamma ray of 700 keV energy Compton scatters from an electron. Find the energy of the photon scattered at 110°, the energy of the scattered ~~photon~~ electron, and the recoil angle of the electron.

Compton scattering equation: $\Delta\lambda = \lambda_c(1-\cos\theta)$

At $\theta = 110^\circ$ $\quad \Delta\lambda = 2.426\times10^{-3}\,\text{nm}\,(1.342)$

$$= 3.256\times10^{-3}\,\text{nm}$$

The initial wavelength of the photon is...

$$\lambda_1 = \frac{hc}{E_1} = \frac{1240\,\text{eV}\cdot\text{nm}}{700\times10^3\,\text{eV}} = 0.001771\,\text{nm}$$

$$= \underline{\underline{1.771\times10^{-3}\,\text{nm}}}$$

The final wavelength of the photon is... $\lambda_2 = \lambda_1 + \Delta\lambda$

$$= 1.771\times10^{-3}\,\text{nm} + 3.256\times10^{-3}\,\text{nm}$$

$$= \underline{5.0274\times10^{-3}\,\text{nm}}$$

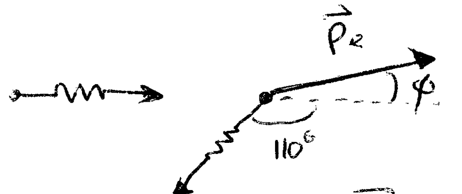
And, the final photon energy is...

$$E_2 = \frac{hc}{\lambda_2} = \frac{1240\,\text{eV}\cdot\text{nm}}{5.0274\times10^{-3}\,\text{nm}} = \boxed{247\,\text{keV}}$$

The electron must receive the difference in energy... $E_{electron} = E_1 - E_2$

This is the electron's kinetic energy $\longrightarrow$ $= \boxed{453\,\text{keV}}$

Use momentum conservation to get the electron recoil angle...



Use the y-component of momentum after the collision: $\quad p_{e_f}\sin\phi = p_{\gamma_f}\sin\theta$

The momentum of the electron can be found from the relativistic relation between Energy and momentum..

$$E^2 = p_e^2 c^2 + (m_e c^2)^2$$

need to use the total energy here... $E_e = K + mc^2 = (453 + 511)\text{ keV}$

$= \underline{\underline{964\text{ keV}}}$

$$(964\text{ keV})^2 = p^2 c^2 + (511\text{ keV})^2$$

or... $p_e c = \sqrt{(964)^2 - (511)^2}\ \text{keV}$

$= \underline{\underline{817\text{ keV}}}$

So...

$$\sin\phi = \frac{p_{\gamma,f}c}{p_{e,f}c}\sin\theta = \frac{E_{\gamma,f}}{p_{e,f}c}\sin\theta$$

$$= \frac{247\text{ keV}}{817\text{ keV}} \cdot \sin(110^\circ)$$

$$\sin\phi = 0.284$$

$$\phi = \sin^{-1}(0.284) = \boxed{16.5^\circ}$$

Additional problem A: Using the spacetime diagram to visualize length contraction.

a). The x'-axis, projected onto the spacetime diagram for frame K lies along a line connecting events with $t'=0$.

Use the Lorentz transformation equation for t'...

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

Set $t'=0$ and find the slope of the relation between ct and x that results..

$$0 = \gamma\left(t - \frac{vx}{c^2}\right) \quad \text{So...}$$

$$t = \frac{vx}{c^2} \qquad \text{or} \qquad \underbrace{ct}_{y\text{-axis}} = \underbrace{\left(\frac{v}{c}\right)}_{\text{slope}} \underbrace{x}_{x\text{-axis}}$$

$$\frac{\Delta(ct)}{\Delta x} = \left(\frac{v}{c}\right)$$

So, the slope of the x'-axis is $\frac{v}{c}$.

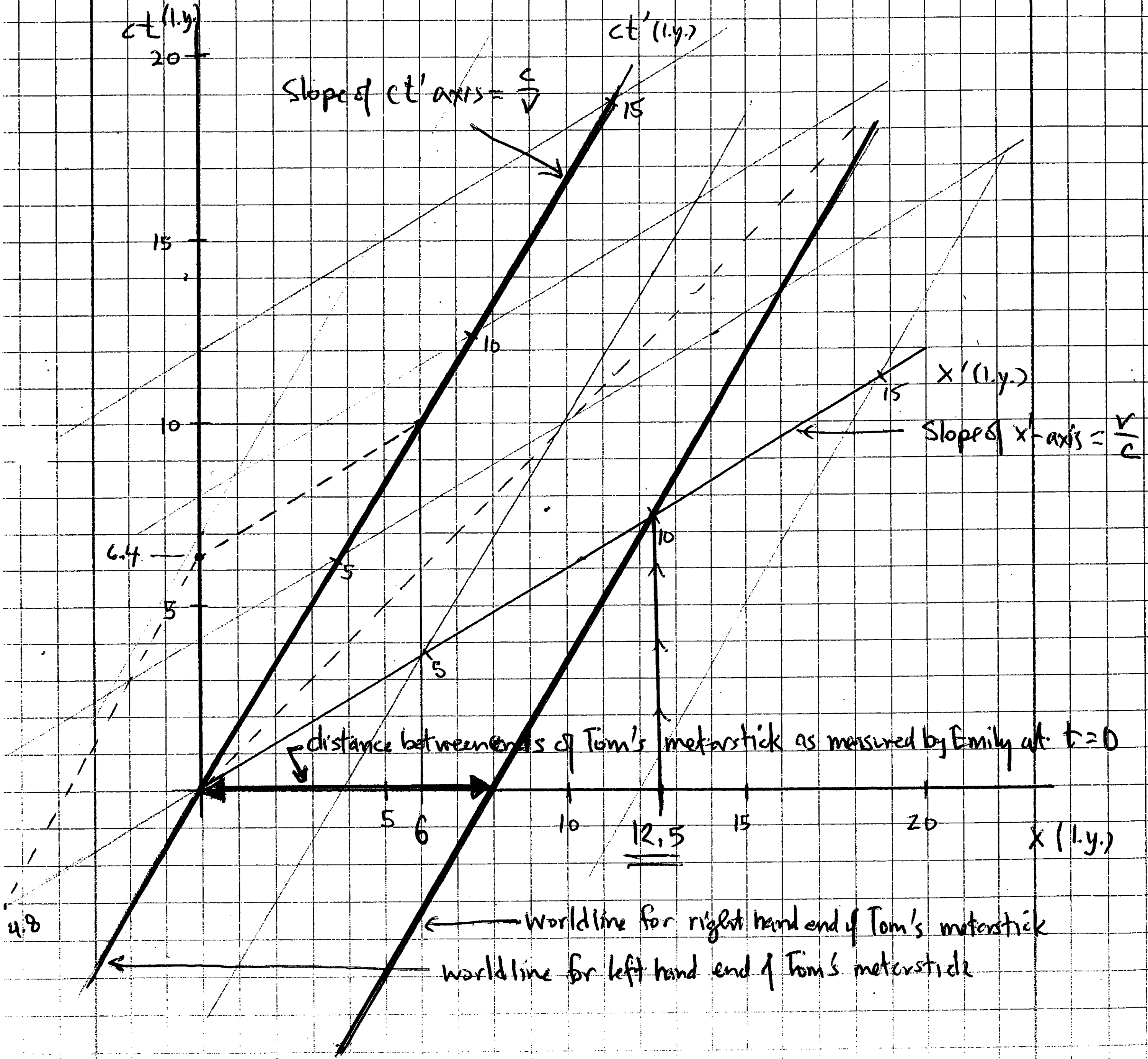
b) To find out how the tick marks should be placed on the x'-axis, consider an event and $t'=0$, $x'=10$ l.y. This event lies on the x'-axis and its location in the frame K is found from the inverse transformation

$$x = \gamma(x' + \underbrace{vt'}_{0}) \rightarrow x = \gamma x' = 1.25 \cdot 10\,\text{l.y.}$$
$$= 12.5\,\text{l.y.}$$

This allows us to place tick marks on the x'-axis,

Twin Paradox : Outbound Leg

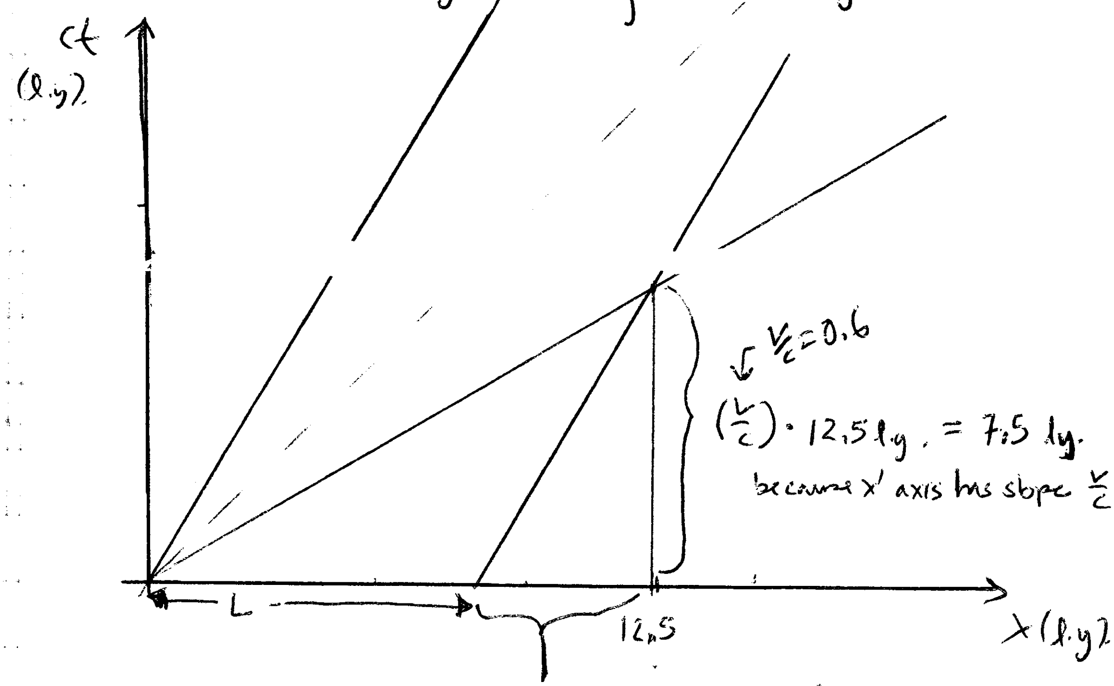
$\frac{v}{c} = 0.6 \quad \gamma = 1.25$



c) To see the length of Tom's meterstick on Emily's spacetime diagram, consider the worldlines for the two ends of Tom's meter stick. These worldlines have a slope of c/v since the meterstick is at rest in Tom's frame (K'). And,... the ~~[illegible]~~ left end of Tom's meterstick has a world line that lies along the ct' axis (if it is placed with its left end at $x'=0$.). The world line for the right end of the meterstick must pass through $x'=10$ l.y.

Emily measures the spatial separation of the two ends of Tom's meterstick simultaneously ... at $t=0$. From the spacetime diagram we see that she gets a result of 8 l.y.

We can work this out geometrically from the diagram.



slope of right end of meterstick $= \frac{c}{v} = \frac{5}{3}$ so this length is $= 7.5$ l.y. $/ \left(\frac{5}{3}\right) = 4.5$ ly

So $L = 12.5 - 4.5 = 8$ l.y.

This is consistent with the length contraction formula $L = L_0\sqrt{1-\left(v/c\right)^2}$

$$= 10\,\text{l.y.}\sqrt{1-\left(\frac{9}{25}\right)} = 8\,\text{l.y.}$$

Additional Problem B: Fill in the missing steps in the derivation of the expression for relativistic kinetic energy.

Eq. 2.57 $\quad K = m\int_0^{\gamma u} u\, d(\gamma u)$

Remember that $\gamma = \dfrac{1}{\sqrt{1-u^2/c^2}}$ here, so this is...

$$K = m\int u\, d\left(\frac{u}{\sqrt{1-u^2/c^2}}\right)$$

This is in a natural form to integrate by parts...

$$\int u\,dv = uv - \int v\,du$$

So...

$$K = m\left[\frac{u^2}{\sqrt{1-u^2/c^2}}\Bigg|_0^u - \int_0^u \frac{u}{\sqrt{1-u^2/c^2}}\,du\right]$$

Change variables to perform this integration.

Let $\eta = 1 - u^2/c^2$, so $d\eta = -\dfrac{2u\,du}{c^2}$

or.. $u\,du = -\dfrac{c^2}{2}\,d\eta$

Then..

$$K = \frac{mu^2}{\sqrt{1-u^2/c^2}} + \frac{mc^2}{2}\int_{\text{[redacted]}}^{1-u^2/c^2} \eta^{-\frac{1}{2}}\,d\eta$$

careful with limits of integration!

$$= \frac{mu^2}{\sqrt{1-u^2/c^2}} + \frac{mc^2}{2}\cdot 2\eta^{\frac{1}{2}}\Big|_{\text{[redacted]}}^{1-u^2/c^2}$$

$$K = \frac{mu^2}{\sqrt{1-u^2/c^2}} + mc^2\left[\sqrt{1-u^2/c^2} - 1\right]$$

now, do some algebra...

$$K = \left[\frac{mu^2}{\sqrt{1 - u^2/c^2}} + mc^2 \sqrt{1 - u^2/c^2} \right] - mc^2$$

Find common denominator...

$$K = \left[\frac{mu^2}{\sqrt{1 - u^2/c^2}} + \frac{mc^2 (1 - u^2/c^2)}{\sqrt{1 - u^2/c^2}} \right] - mc^2$$

$$K = \frac{mc^2}{\sqrt{1 - u^2/c^2}} - mc^2$$

$$\boxed{K = mc^2 \left[\frac{1}{\sqrt{1 - u^2/c^2}} - 1 \right]}$$

which is Eq 2.58

Additional Problem C: Relativistic freefall

Newton's 2nd Law $\quad F = \frac{dp}{dt}$.

If F = constant..., $= F_0$ then, starting from $p_i = 0$....

$$p(t) = \int_0^t F\,dt = F_0 t$$

But we must use relativistic momentum so...

$$\frac{mu}{\sqrt{1-u^2/c^2}} = F_0 t \qquad \text{solve for } u:$$

$$mu = F_0 t \sqrt{1-u^2/c^2} \qquad \text{divide by } m \text{ and square both sides}$$

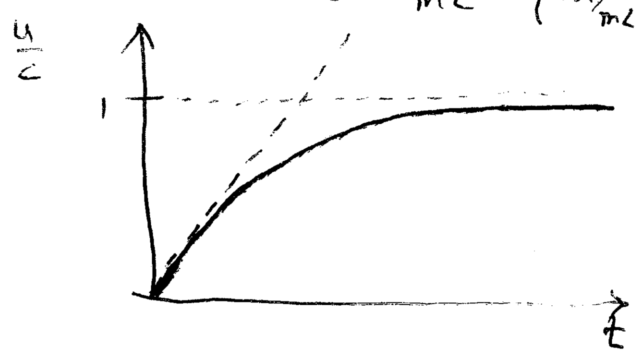
$$u^2 = \left(\frac{F_0 t}{m}\right)^2 \left(1 - u^2/c^2\right)$$

$$u^2\left(1 + \left(\frac{F_0 t}{mc}\right)^2\right) = \left(F_0 t/m\right)^2$$

$$\boxed{u = \frac{F_0 t}{m} \frac{1}{\sqrt{1+\left(F_0 t/mc\right)^2}}} \quad \text{or} \quad \boxed{\frac{u}{c} = \frac{F_0 t}{mc} \cdot \frac{1}{\sqrt{1+\left(F_0 t/mc\right)^2}}}$$

Newtonian result...

Limit as $t \to \infty$... $\quad \frac{u}{c} = \frac{F_0 t}{mc} \cdot \frac{1}{\left(F_0 t/mc\right)} = 1$



Velocity approaches c ... cosmic speed limit.