Introduction to Power Series

Basic definition

A series

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

is called a *power series*. The domain of the power series function is the set of all x values for which the series converges.

Here is a simple example to demonstrate that in the typical power series you will have convergence for some values of x and divergence for others.

$$\sum_{n=0}^{\infty} x^n$$

This series is quite clearly a geometric series, and converges for $|x| < 1$.

Another example is the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (n!)^2}$$

We can use the ratio test to determine where this series converges.

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{\dfrac{(-1)^{n+1} x^{2n+2}}{2^{2n+2} ((n+1)!)^2}}{\dfrac{(-1)^n x^{2n}}{2^{2n} (n!)^2}} \right| = \lim_{n \to \infty} \left| \frac{x^2}{2^2 (n+1)^2} \right| = 0$$

This series converges for all x.

Power series centered at a

A basic variant of the power series is the power series centered at $x = a$.

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$$

For example,

$$\sum_{n=0}^{\infty} \frac{(x-1)^n}{n}$$

is a power series centered at $x = 1$. To determine where this series converges we can apply the ratio test.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(x-1)^{n+1}}{n+1}}{\frac{(x-1)^n}{n}}\right| = \lim_{n\to\infty} \left|(x-1)\frac{n}{n+1}\right| = |x-1|$$

By the ratio test, this series converges whenever $|x-1| < 1$ and diverges when $|x-1| > 1$.

Radius of convergence

The last example is typical of the behavior of power series. For a power series centered at a there is a number R called the *radius of convergence* of the power series such that the series converges when $|x-a| < R$ and diverges when $|x-a| > R$. Note that in some cases R may be $+\infty$. Every example we have seen above has an associated radius of convergence.

1. $\sum_{n=0}^{\infty} x^n$ is a power series about $x = 0$ with $R = 1$.

2. $\sum_{n=0}^{\infty} (-1)^n x^{2n}/2^{2n}(n!)^2$ is a power series about $x = 0$ with $R = +\infty$.

3. $\sum_{n=0}^{\infty} (x-1)^n/n$ is a power series about $x = 1$ with $R = 1$.

Connecting power series to known functions

Many power series turn out to be alternative ways to represent functions we are already familiar with.

For example, since the power series

$$\sum_{n=0}^{\infty} x^n$$

we can sum it and see that

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

(Note that the inequality is only true for those values of x for which the series converges.)

We can connect powers series to a larger set of known functions by use of the following

Theorem If the power series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

has a radius of convergence R the function $f(x)$ is differentiable everywhere in the interval $(a-R, a+R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n\, c_n (x-a)^{n-1} \tag{1}$$

The function $f(x)$ is also integrable everywhere in the interval $(a-R, a+R)$ and

$$\int f(x)\, dx = \sum_{n=0}^{\infty} \frac{c_n}{n+1} (x-a)^{n+1} \tag{2}$$

The series in (1) and (2) both have radius of convergence R.

This theorem can be used to broaden the range of examples for which a power series can be connected to a known function. Here are some examples.

$$f(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$f'(x) = \sum_{n=1}^{\infty} n\, x^{n-1} = \left(\frac{1}{1-x}\right)' = \frac{-1}{(1-x)^2}$$

$$\int f(x)\, dx = \sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = \int \frac{1}{1-x} dx = -\ln|1-x|$$

$$g(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n} = \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1+x^2}$$

$$\int g(x)\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \int \frac{1}{1+x^2}\,dx = \tan^{-1} x$$

$$h(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$h'(x) = \sum_{n=0}^{\infty} n \frac{x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

The last calculation suggests that $h(x) = e^x$, because e^x is another function we know that is equal to its own derivative.

These examples are starting to suggest that many of the functions we know in mathematics are related to power series. In the next section we will make this connection more explicit and show that all of the standard functions in mathematics have equivalent power series.