The geometric series and the ratio test

Today we are going to develop another test for convergence based on the interplay between the limit comparison test we developed last time and the geometric series.

A note about the geometric series

Before we get into today's primary topic, I have to clear up a little detail about the geometric series. Here is a formula for the geometric series.

$$\sum_{n=0}^{\infty} a\, r^n = \frac{a}{1 - r}$$

An important detail to note here is that the sum starts with $n = 0$. Many times in what follows we will find ourselves having to look at variants of the geometric series that start at an index other than 0. These cases are very easy to handle.

$$\sum_{n=N}^{\infty} a\, r^n = \sum_{n=N}^{\infty} a\, r^{n-N}\, r^N = \sum_{k=0}^{\infty} a\, r^k\, r^N = r^N \sum_{k=0}^{\infty} a\, r^k = r^N \frac{a}{1 - r}$$

The one special trick needed here was a change in index. In going from the second series to the third, we replaced the index n with a new index k where $k = n - N$.

1 The ratio test

Both the comparison test and the limit comparison test have a major drawback; namely, to use those tests to understand some series we have to be able to compare that series to something we already understand. In many cases this is difficult to do. Perhaps the prime example of this is the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

The presence of the factorial term makes it difficult to compare this series with any other know series (or even to do an integral comparison). What we need to handle this case is some notion of convergence that exist independently of comparisons.

Rethinking convergence

What does a convergent series look like? For starters, we know that if

$$\sum_{n=1}^{\infty} a_n$$

is to have any chance of converging, its terms must get small. However, as we have seen, merely requiring

$$\lim_{n \to \infty} a_n = 0$$

is not sufficient. There are many examples of divergent series that do this, so the requirement that the terms get small as n gets large is not conclusive.

For convergence to happen, the first requirement is that terms in the series have to get small. What we need is a slightly more clever way to express the idea that the terms are getting small. Here is that clever way to characterize the fact that the terms get smaller.

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1 \tag{1}$$

You have to stare for a while to convince yourself that this expresses the fact that the terms in series are getting smaller as n gets larger.

Lets take as given that the limit (1) exists and is less than 1. Pick an ϵ with the property that

$$L + \epsilon < 1$$

If we go far enough out in the sequence of terms, the thing we are taking the limit of will stay smaller than $L + \epsilon$.

$$\text{For } n > N \text{ we have } \left| \frac{a_{n+1}}{a_n} \right| < L + \epsilon < 1$$

We can rewrite

$$\left| \frac{a_{n+1}}{a_n} \right| < L + \epsilon$$

as

$$a_{n+1} < (L + \epsilon)\, a_n \tag{2}$$

Note that this is only legal if all the terms in the series were positive to begin with. Here is how we can use the inequality (2) to get something useful.

$$a_{n+1} < (L+\epsilon)\, a_n$$

$$a_{n+2} < (L+\epsilon)\, a_{n+1} < (L+\epsilon)\,(L+\epsilon)\, a_n$$

More generally we get that

$$a_{n+k} < (L+\epsilon)^k\, a_n$$

This all allows you to say the following:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n$$

$$< \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} (L+\epsilon)^{n-(N+1)}\, a_{N+1}$$

$$= \sum_{n=1}^{N} a_n + a_{N+1} \sum_{n=N+1}^{\infty} (L+\epsilon)^{n-(N+1)}$$

The last sum that shows up here is the geometric series, and it shows that this whole thing converges.

The reasoning above leads us to the following

Theorem Let $\sum_{n=1}^{\infty} a_n$ be a series with only positive terms with the property that

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1$$

Then the series must converge.

There is a closely related proof forthe divergent case, which leads to

Theorem Let $\sum_{n=1}^{\infty} a_n$ be a series with only positive terms with the property that

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1$$

Then the series must diverge.

A word of warning

The ratio test is very handy, and easy to apply, but there is one case in which it does not provide useful information. The test says that when the limit

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

is less than 1 the series will converge, and when the limit is greater than 1 the series will diverge. If the limit is exactly one, this test tells us nothing.

Here is an example to demonstrate this. Consider the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

We know that this series diverges. Applying the ratio test gives us

$$\lim_{n\to\infty} \frac{1/(n+1)}{1/n} = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

Now consider convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

In this case the ratio limit is

$$\lim_{n\to\infty} \frac{1/(n+1)^2}{1/n^2} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$$

These two examples demonstrate that the when the ratio limit is one the ratio test is completely inconclusive: the series could either converge or diverge.

Example

The ratio test now allows to do some examples that would have been very difficult before. The best such example is the following.

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

$$\lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n\to\infty} \frac{2^{n+1}\, n!}{2^n\,(n+1)!} = \lim_{n\to\infty} 2\, \frac{n!}{(n+1)\, n!} = \lim_{n\to\infty} \frac{2}{n+1} = 0$$

This shows that the series converges.

This example demonstrates that the ratio test is especially well suited to handle series containing exponentials or factorials, as those things simplify particularly well when we take a ratio.

Here is a second example

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

$$\lim_{n \to \infty} \frac{(n+1)/2^{n+1}}{n/2^n} = \lim_{n \to \infty} \frac{2^n\,(n+1)}{2^{n+1}\;n} = \lim_{n \to \infty} \frac{n+1}{2\,n} = \frac{1}{2}$$

This series also converges.

2 The nth-root test

As powerful and useful as the ratio test is, there are certain examples that are so nasty that something even stronger is required to solve them. Consider the example

$$\sum_{n=1}^{\infty} \frac{3^n}{n^n}$$

Applying the ratio test means having to compute the limit

$$\lim_{n \to \infty} \frac{3^{n+1}/(n+1)^{n+1}}{3^n/n^n} = ?$$

This is not a particularly easy limit to compute.

A powerful test which often helps in cases like this one is the nth root test. I am not going to show the proof of the validity of the nth root test, but the methods of proof are very similar to those I used above to prove part of the ratio test.

Theorem (nth root test) Let $\sum_{n=1}^{\infty} a_n$ be a series with only positive terms with the property that

$$\lim_{n \to \infty} \sqrt[n]{a_n} < 1$$

Then the series converges.

Theorem (nth root test) Let $\sum\limits_{n=1}^{\infty} a_n$ be a series with only positive terms with the property that

$$\lim_{n \to \infty} \sqrt[n]{a_n} > 1$$

Then the series diverges.

When applied to the example at the start of this section

$$\sum_{n=1}^{\infty} \frac{3^n}{n^n}$$

the nth root test quickly and easily shows that the series converges.

$$\lim_{n \to \infty} \sqrt[n]{\frac{3^n}{n^n}} = \lim_{n \to \infty} \frac{3}{n} = 0$$