Series Comparisons

1 Simple series comparisons

The next theorems demonstrate how to use known series to learn things about closely related examples.

Theorem If $\sum_{n=0}^{\infty} a_n$ converges and $0 \leq b_n \leq a_n$ for all n, then $\sum_{n=0}^{\infty} b_n$ converges, too.

Theorem If $\sum_{n=0}^{\infty} a_n$ diverges and $b_n \geq a_n \geq 0$ for all n, then $\sum_{n=0}^{\infty} b_n$ diverges, too.

Example Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n\ 2^n}$$

We can understand this series by making a comparison with the known, convergent geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} r^n = \frac{1}{1 - r} = \frac{1}{1 - 1/2} = 2$$

Because

$$0 < \frac{1}{n\ 2^n} < \frac{1}{2^n}$$

for all n, we see that the series converges, and we even get a crude estimate for its value.

$$0 < \sum_{n=1}^{\infty} \frac{1}{n\ 2^n} < 2$$

In order to apply this method, we need to have a list of “known” series to use as the basis for our comparisons. These are the things that we know:

$$\sum_{n=1}^{\infty} r^n = \frac{1}{1 - r} \ for\ |r| < 1$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{converges} & p > 1 \\ \text{diverges} & p \leq 1 \end{cases}$$

In other cases, you can use the integral comparison test to understand some "simple" series, and then use series comparison from there to tell you things.

2 An example to motivate further developments

Next we are going to see an example which demonstrates that although the comparison test is powerful, it is often difficult to apply.

Consider the series

$$\sum_{n=1}^{\infty} \frac{n+1}{(n+2)\,n}$$

The first thing you should notice about this series is that it strongly resembles the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

which we know to diverge. That part is easy. The difficulty comes when you notice that the most simple and obvious comparison you can write down is not at all useful.

$$\sum_{n=1}^{\infty} \frac{n+1}{(n+2)\,n} < \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

Saying that a series is less than a series which is known to diverge is not useful information. It tells us nothing about the series in question. The way out of this impasse is to concoct a trickier comparison which does tell us something useful. Consider the comparison

$$\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} < \sum_{n=1}^{\infty} \frac{n+1}{(n+2)\,n}$$

The comparison is based on the observation that

$$\frac{1}{2} < \frac{n+1}{n+2}$$

for all $n \geq 1$. The comparison is a useful one, because $1/2$ times the harmonic series diverges just as surely as the harmonic series.

$$\infty = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n} < \sum_{n=1}^{\infty} \frac{n+1}{(n+2)\, n}$$

Thus we see that the original series diverges.

This example demonstrates that although the comparison test is useful, it often requires inordinate cleverness to apply correctly.

3 A simpler method based on comparison

Next we are going to develop a technique based on the comparison method, but which does not require nearly as much cleverness. I am going to make an effort to manufacture this method from first principles. This is a valuable exercise, in that it should give you some valuable insight into the workings of this method and other methods which are coming down the line.

We begin the discussion with a vague but useful

Basic idea If a series 'looks like' something that converges or diverges, the series must also converge or diverge, respectively.

The most obvious question that this statement raises is "What does it mean for one series to 'look like' another?" Here is one possible way to express this concretely. Saying

$$\sum_{n=1}^{\infty} a_n \text{ looks like } \sum_{n=1}^{\infty} b_n$$

basically means

$$a_n \text{ looks like } b_n$$

or

$$\left| b_n - a_n \right| \text{ is about } 0$$

More concretely, you can state this in terms of a limit.

$$\lim_{n \to \infty} \left| b_n - a_n \right| = 0$$

To make this more concrete, suppose that we have two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ and we know the second series converges and further

$$\lim_{n \to \infty} \left| b_n - a_n \right| = 0$$

Can we conclude that the first series converges, too? Is this vague idea good enough to make a theorem?

Unfortunately, the answer is no. If you think about it for a while, you can probably come up with an example like the following. Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

which is known to converge. Now consider the second series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Note that

$$\lim_{n \to \infty} \left| b_n - a_n \right| = \lim_{n \to \infty} \left| \frac{1}{n^2} - \frac{1}{n} \right| = 0$$

The supposed theorem would have us conclude that the second series converges. This is clearly not the case, so our claim is not true.

Can we salvage anything from this failure? Is there another way to express the notion that one series ‘looks like’ another series? Here is one other way

$$\lim_{n \to \infty} \left| \frac{a_n}{b_n} \right| = 1$$

The idea is that two series ‘look like’ each other if the ratio of their terms approaches 1 as n gets large.

To make this more concrete, suppose that we have two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ and we know the second series converges and further

$$\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = 1$$

Is it a theorem that the first series must also converge? Again, we might try this with an example first.

$$\sum_{n=1}^{\infty}\frac{n+1}{(n+2)\,n^2} \text{ compared to } \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\lim_{n\to\infty}\left|\frac{\dfrac{n+1}{(n+2)\,n^2}}{\dfrac{1}{n^2}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n+2}\right| = 1$$

The theorem would have us believe that the first series converges. This is actually the case, as you can see via the comparison

$$\sum_{n=1}^{\infty}\frac{n+1}{(n+2)\,n^2} < \sum_{n=1}^{\infty}\frac{1}{n^2}$$

Can we actually prove that this stuff really works?

4 The limit comparison theorem

Theorem (limit comparison test) If the series $\sum_{n=1}^{\infty} b_n$ converges and

$$\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = k \tag{1}$$

then the series $\sum_{n=1}^{\infty} a_n$ converges, too.

Proof (based on the comparison test) We are going to try to show that given the limit (1) we can get

$$\sum_{n=1}^{\infty} a_n < (\text{something else}) \sum_{n=1}^{\infty} b_n$$

and that the series on the right converges

What is the fudge factor that I need in order to get that comparison to go? How do I get from the only other thing that I know,

$$\lim_{n \to \infty} \left|\frac{a_n}{b_n}\right| = k$$

Here is how you make that happen.

$$\lim_{n \to \infty} \left|\frac{a_n}{b_n}\right| = k$$

says to you that as soon as n gets big enough, the ratio gets ‘arbitrarily close’ to k. The way to express that concretely is to say that given some factor λ which is very small, there is an N such that for all $n > N$ we have

$$k - \lambda < \left|\frac{a_n}{b_n}\right| < k + \lambda$$

The following steps demonstrate how that inequality can help us.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} a_n = \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} b_n \frac{a_n}{b_n}$$

$$< \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} b_n \left|\frac{a_n}{b_n}\right|$$

$$< \sum_{n=1}^{N} a_n + \sum_{n=N+1}^{\infty} b_n \, (k + \lambda)$$

$$< \sum_{n=1}^{N} a_n + (k + \lambda) \sum_{n=N+1}^{\infty} b_n$$

Since the b_n series converges, everything in the last line converges. We have thus shown that the original series converges, because we have trapped it underneath some terms which are known to be finite. Thus the theorem is proved.

There is a similar version of this theorem for divergent series. I leave it to you to look up the statement of that theorem in the text.

Let's bring back the original example now

$$\sum_{n=1}^{\infty} \frac{n+1}{(n+2)\,n} \tag{2}$$

Compare this to

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

via the limit

$$\lim_{n \to \infty} \left| \frac{\dfrac{n+1}{(n+2)\,n}}{1/n} \right| = 1$$

Since the limit is 1, the divergent version of the theorem shows that the series (2) diverges. Note that we accomplished this without any of the cleverness needed to apply the raw comparison test.