Series with positive and negative terms

1 Alternating Series

An alternating series is a series in which the signs on the terms being added alternate between + and -. Here is perhaps the most famous alternating series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

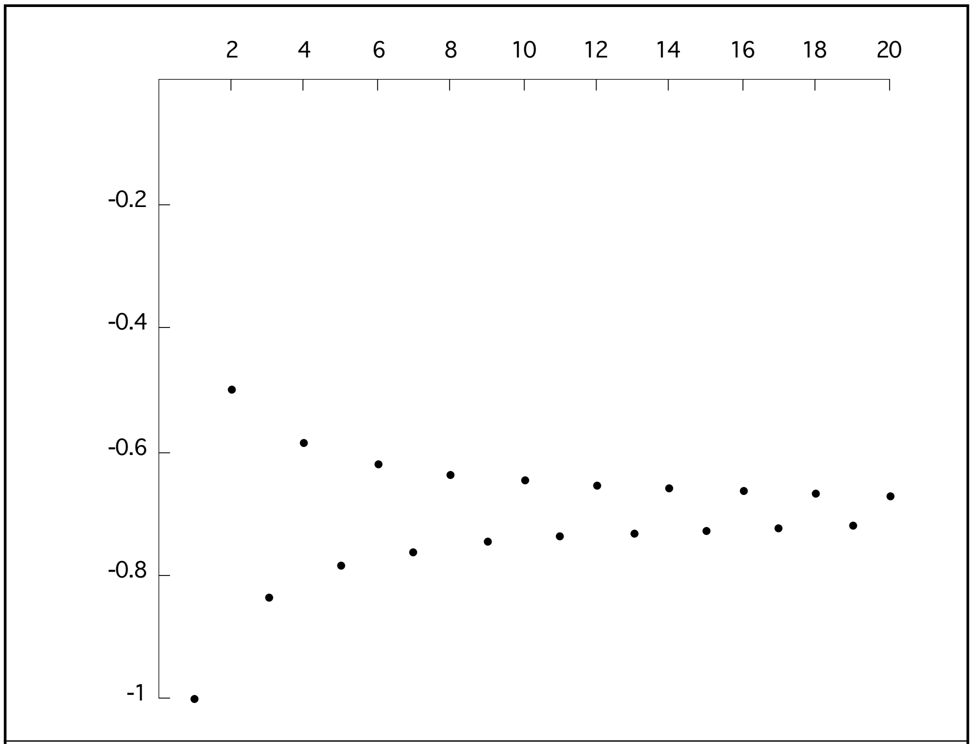
This series actually converges. Let us now show this. We have to start from first principles by examining the partial sum.

$$S_N = \sum_{n=1}^{N} \frac{(-1)^n}{n}$$

What is $\lim_{N\to\infty} S_N$? We can get a sense for what happens by making a table of partial sums.

N	S_N
1	-1
2	-0.5
3	-0.83333333333333326
4	-0.58333333333333326
5	-0.78333333333333321
6	-0.6166666666666667
7	-0.7595238095238096
8	-0.63452380952380938
9	-0.74563492063492043
10	-0.64563492063492056
11	-0.7365440115440115
12	-0.65321067821067802
13	-0.73013375513375489
14	-0.65870518370518349
15	-0.72537185037185015
16	-0.66287185037185015
17	-0.72169537978361487
18	-0.66613982422805929
19	-0.71877140317542765
20	-0.66877140317542783

The table is not very illuminating, but if we plot the partial sums as a function of N, something rather dramatic jumps out at you.



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To try to understand this picture, let us take a closer look at the nature of the partial sum.

$$S_N = \sum_{n=1}^{N} \frac{(-1)^n}{n}$$

To start with, write down a typical partial sum for even N. Grouping the terms in that partial sum in a particular way leads to an interesting observation.

$$S_6 = \left(-1 + \frac{1}{2}\right) - \left(\frac{1}{3} - \frac{1}{4}\right) - \left(\frac{1}{5} - \frac{1}{6}\right) = \frac{-1}{2} - (\text{pos}) - (\text{pos})$$

This tells us that the sequence of even partial sums is a decreasing sequence which starts at -1/2. This raises the question of whether that sequence of partial sums will decrease to $-\infty$ or if it will level out and have a limit. If we can somehow show that this sequence of partial sums is bounded below, then we will have shown that the sequence converges.

Set aside that question for a moment and examine a typical partial sum with odd N. Once again, an appropriate version of the grouping trick shows something interesting.

$$S_7 = -1 + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{6} - \frac{1}{7}\right)$$

$$= -1 + (\text{pos}) + (\text{pos}) + (\text{pos})$$

The odd partial sums form an increasing sequence of terms. The final question to answer is "what is the relationship between the even partial sums and the odd partial sums?"

How does, for example, S_7 compare to all of the even partial sums that come after it?

$$S_{2\,k} = S_7 + \sum_{n=8}^{2k} \frac{(-1)^n}{n}$$

$$\sum_{n=8}^{2k} \frac{(-1)^n}{n} = \left(\frac{1}{8} - \frac{1}{9}\right) + \left(\frac{1}{10} - \frac{1}{11}\right) + \ldots + \frac{1}{2\,k} > 0$$

$$S_{2\,k} = S_7 + \text{pos}$$

$$S_{2\,k} > S_7$$

What we have now is that the sequence of even partial sums is a strictly decreasing sequence which is bounded below, and therefore must converge.

By similar reasoning, the story with the odd partial sums is that they are strictly increasing and bounded above. Hence, they must converge.

There is just one question left to resolve, and here the picture is not much help. We have established that separately the even partial sums and the odd partial sums both form convergent sequences. The only thing left to prove is that these two limits are in fact the same limit. We can do that pretty easily by examining the difference between the even and odd partial sums in the limit.

$$\lim_{k\to\infty} \left|S_{2k} - S_{2k+1}\right| = \lim_{k\to\infty} \left|\frac{-(-1)^{2k+1}}{(2k+1)}\right| = \lim_{k\to\infty} \frac{1}{2\,k+1} = 0$$

This finally shows that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges.

If you go back and take a close look at what we have just shown, you will see that the entire proof hangs on one essential fact about $(-1)^n/n$: in absolute value each term is smaller than the term that came before it. Therefore, if we have any series with exactly alternating signs and strictly decreasing terms, we can apply exactly the same arguments to it to show that the series converges.

Theorem (Alternating series test) If the terms of the series $\sum_{n=1}^{\infty} (-1)^n a_n$ have the property that all of the a_n terms are positive and

$$a_{n+1} < a_n$$

for all n, then the series converges.

2 Absolute Convergence

Alternating series are very nice. To prove that an alternating series converges, we only have to prove that the terms in the series decrease in absolute value as n increases. A practical problem that sometimes occurs in practice is that you may have a series that has both positive and negative terms in it but the positive and negative terms do not follow the strict alternating pattern. An example of this kind is the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$$

This series has both positive and negative terms, but the terms don't form a strict alternation. In cases like this, the only thing we can do is the following.

Definition A series

$$\sum_{n=1}^{\infty} a_n$$

is *absolutely convergent* if the series

$$\sum_{n=1}^{\infty} |a_n|$$

converges. A series is *conditionally convergent* if the series converges, but not absolutely.

The series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$$

is absolutely convergent because

$$\sum_{n=1}^{\infty}\left|\frac{\cos n}{n^2}\right| < \sum_{n=1}^{\infty}\frac{1}{n^2}$$

and the latter series is known to converge.