

Series

Definition of Series

A series is an infinite sum

$$\sum_{k=0}^{\infty} a_k$$

Series are closely related to sequences, because one of the standard techniques used to understand series is to introduce the sequence of *partial sums*.

$$s_n = \sum_{k=0}^n a_k$$

In terms of the partial sum sequence,

$$\sum_{k=0}^{\infty} a_k = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k = \lim_{n \rightarrow \infty} s_n$$

If the sequence of partial sums converges to a limit, we say that the original series converges to that limit. If the sequence of partial sums diverges, we say that the series diverges.

This device for understanding series is especially useful in developing the theory of series, although it is of only limited practical usefulness. The problem with most series is that it is very difficult to compute what

$$s_n = \sum_{k=0}^n a_k$$

is as a function of n .

Three important examples

There are a few isolated cases where it is possible to compute a partial sum by some means. In this section of the lecture we will encounter three such examples.

Here is the most important example, the geometric series

$$\sum_{k=0}^{\infty} r^k$$

The key to understanding the partial sums of the geometric series is the algebraic identity

$$r^{n+1} - 1 = (r - 1)(1 + r + r^2 + r^3 + \dots + r^n)$$

Using this identity we can show that

$$s_n = \sum_{k=0}^n r^k = 1 + r + r^2 + r^3 + \dots + r^n = \frac{r^{n+1} - 1}{r - 1}$$

This gives us an explicit formula for the partial sum s_n . With that formula we can compute the sum of the series as the limit of partial sums.

$$\sum_{k=0}^{\infty} r^k = \lim_{n \rightarrow \infty} \sum_{k=0}^n r^k = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{r^{n+1} - 1}{r - 1}$$

Provided that $|r| < 1$, the latter limit exists and we get that

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1 - r}$$

Note that if $r \geq 1$ the original sum diverges.

The next example is

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

This example can be handled by a trick: we use partial fractions to rewrite the partial sum as

$$s_n = \sum_{k=1}^n \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

If you write out the first few terms of a partial sum, you will see that almost all of the terms cancel.

$$s_4 = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) = 1 - \frac{1}{5}$$

In general, what happens here is that

$$s_n = 1 - \frac{1}{n+1}$$

Thus

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(1 - \frac{1}{n+1} \right) = 1$$

The final example is the *harmonic series*

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

It is possible to show that the harmonic series diverges by examining a carefully constructed sequence of partial sums.

$$s_1 = 1$$

$$s_2 = 1 + \frac{1}{2}$$

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4} \right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4} \right) = 1 + \frac{2}{2}$$

$$s_8 = s_4 + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) > s_4 + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \right) > 1 + \frac{3}{2}$$

The same pattern continues through all the powers of 2. Every time we add more terms to the partial sum it continues to grow. Thus, in the limit as n gets very large s_n diverges.

A necessary condition for convergence

If a sequence of partial sums is going to have any chance to converge, the terms in the series have to get smaller as the summation index k gets larger. However, as the last two examples above demonstrate, simply having the terms of the series get smaller as the index k grows is not strong enough to guarantee convergence. In the case of the series

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

the individual terms $1/k(k+1)$ shrink in size as k gets large and the series converges. However, in the case of the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

the individual terms $1/k$ shrink as k gets larger, but the series itself diverges.

We can summarize this state of affairs by saying that

$$\lim_{k \rightarrow \infty} a_k = 0$$

is a *necessary condition* for convergence of the series

$$\sum_{k=0}^{\infty} a_k$$

but not a *sufficient condition* for convergence.

Combining series

Except for the complication caused by the fact that the upper limit in the summation

$$\sum_{k=0}^{\infty} a_k$$

is ∞ , a series is basically a sum. All of the familiar algebraic rules that you know about sums like the distributive law continue to hold for series (as long as all of the series involved are convergent series). Here are some algebraic laws that we are going to make use of.

$$\sum_{k=0}^{\infty} c a_k = c \sum_{k=0}^{\infty} a_k$$

$$\sum_{k=0}^{\infty} (a_k + b_k) = \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k$$

$$\sum_{k=0}^{\infty} (a_k - b_k) = \sum_{k=0}^{\infty} a_k - \sum_{k=0}^{\infty} b_k$$