Introduction to Sequences

A sequence a_n is a list of numbers indexed by some index n. Usually when we talk about sequences we express them as a rule that tells us how to generate the n^{th} sequence element a_n from the index value n, starting from some $n = n_0$. Here are some concrete examples.

$$a_n = \frac{n+1}{n} \; ; n \geq 2$$

$$a_n = \frac{1 + (-1)^n}{2^n} \; ; n \geq 0$$

$$a_n = \frac{n^2 + 1}{n^3 + 1} \; ; n \geq 1$$

Once we know the rule for generating the n^{th} sequence element, we can write out the first few elements of the sequence. For example, the first few terms of the third example above are

$$1, 5/9, 5/14, 17/65, 13/63, 37/217, ...$$

Given just the sequence of terms written out like this it may be extremely difficult to recover the rule that generated that sequence. Fortunately, in almost every case we will encounter we will have that rule.

The limit of a sequence

The most fundamental question to ask about a sequence is whether or not it has a *limit* as the index n gets large. To answer this question in an informal way, we could look at the first few terms of the sequence and then make a conjecture about where the sequence is going as the index grows. Here are the first ten terms of each of the three sequences shown above.

$$\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \frac{8}{7}, \frac{9}{8}, \frac{10}{9}, \frac{11}{10}, \frac{12}{11}, ...$$

$$1, 0, \frac{1}{2}, 0, \frac{1}{8}, 0, \frac{1}{32}, 0, \frac{1}{128}, 0, ...$$

$$1, \frac{5}{9}, \frac{5}{14}, \frac{17}{65}, \frac{13}{63}, \frac{37}{217}, \frac{25}{172}, \frac{65}{513}, \frac{41}{365}, \quad \frac{101}{1001}, ...$$

From examining the first few terms of a sequence we can sometimes make a conjecture about what that sequence of terms is going to do in the limit as the index grows very large. In each of the cases we are looking at here, we can make reasonable guesses about what happens in the limit as n gets large.

$$\lim_{n \to \infty} \frac{n+1}{n} = 1$$

$$\lim_{n \to \infty} \frac{1 + (-1)^n}{2^n} = 0$$

$$\lim_{n \to \infty} \frac{n^2 + 1}{n^3 + 1} = 0$$

A formal definition

Since we are about to start generating theorems about the behavior of sequences, we have to establish a definition for the limit of a sequence.

Definition We say that the sequence a_n *converges to a limit* L if, given any positive number ϵ there is a whole number N with the property that $|a_n - L| < \epsilon$ for all $n \geq N$.

Proving that a sequence converges directly from the definition is challenging. Here is a simple example. One of our examples earlier was the sequence

$$a_n = \frac{n+1}{n}$$

We conjectured that this sequence has a limit of 1 as n goes off to ∞. Here is how to go about proving that from the definition:

- Suppose that someone has given you a very small positive number ϵ.

- You want to show that as n gets large enough, the difference between a_n and $L = 1$ is smaller than ϵ:

$$|a_n - L| = \left|\frac{n+1}{n} - 1\right| < \epsilon$$

$$\left|\frac{(n+1) - n}{n}\right| < \epsilon$$

$$\frac{1}{n} < \epsilon$$

$$n > \frac{1}{\epsilon}$$

- The calculation shows that if we pick N to be the smallest integer larger than $1/\epsilon$, then $n \geq N$ implies that $\left|a_n - L\right| < \epsilon$. This demonstrates that the sequence a_n converges and that its limit is 1.

Here is another very useful example. Let r be any positive number less than 1. The sequence

$$a_n = r^n$$

converges to 0 as n gets large:

$$\left|a_n - L\right| = \left|r^n - 0\right| < \epsilon$$

$$r^n < \epsilon$$

$$n \ln r < \ln \epsilon$$

$$n > \frac{\ln \epsilon}{\ln r}$$

Some helpful theorems

Working with the definition and using the definition to prove that a particular sequence converges quickly becomes very challenging as soon as you move beyond the most basic examples. The most effective method to prove that a particular sequence has a limit is to use one of the following three theorems.

Theorem One: Combinations of Sequences Let a_n and b_n be convergent sequences with

$$\lim_{n \to \infty} a_n = A \text{ and } \lim_{n \to \infty} b_n = B \neq 0$$

then

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = A + B$$

$$\lim_{n\to\infty} (a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = A - B$$

$$\lim_{n\to\infty} a_n\, b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = A\ B$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{A}{B}$$

The most common way we use this theorem will be to 'break down' sequences into simpler component parts. Provided that we can ultimately show that those component parts converge, the process is justified.

$$\lim_{n\to\infty} (1/2)^n + \frac{(1/3)^n}{4} =$$

$$\lim_{n\to\infty} (1/2)^n + \lim_{n\to\infty} \frac{(1/3)^n}{4} =$$

$$\lim_{n\to\infty} (1/2)^n + \frac{\lim_{n\to\infty} (1/3)^n}{\lim_{n\to\infty} 4} =$$

$$0 + \frac{0}{4}$$

In the last step we used the fact that the sequence $a_n = r^n$ converges to 0 whenever r is less than 1.

Theorem Two: Squeeze Theorem Let a_n and b_n be convergent sequences with

$$\lim_{n\to\infty} a_n = L \text{ and } \lim_{n\to\infty} b_n = L$$

and suppose further that c_n is a sequence and that for all $n > n_0$ we have

$$a_n \le c_n \le b_n$$

then the sequence c_n converges to L.

We can use the squeeze theorem to prove that the sequence

$$a_n = \frac{1 + (-1)^n}{2^n}$$

converges to 0. It is easy to show that

$$0 \le \frac{1 + (-1)^n}{2^n} \le \frac{2}{2^n} = 2 \left(\frac{1}{2}\right)^n$$

Both the sequence consisting of all 0s and the sequence on the right converge to 0, so our sequence is forced to converge to 0 along with them.

Theorem Three: Comparison Theorem Suppose $f(x)$ is a function with the property that

$$\lim_{x \to \infty} f(x) = L$$

and that $a_n = f(n)$. Then the sequence a_n converges to L.

This theorem provides an easy way to prove that the sequence

$$a_n = \frac{n^2 + 1}{n^3 + 1}$$

converges and has a limit of 0. Consider the function

$$f(x) = \frac{x^2 + 1}{x^3 + 1}$$

We can use L$'$Hospital's rule to compute the limit of $f(x)$ as x goes to ∞:

$$\lim_{x \to \infty} \frac{x^2 + 1}{x^3 + 1} = \lim_{x \to \infty} \frac{2\,x}{3\,x^2} = \lim_{x \to \infty} \frac{2}{3\,x} = 0$$

Since $a_n = f(n)$ for this particular function, we have a proof that the limit of a_n is 0 as n goes to ∞.

Sequences that diverge to infinity

Another very common sort of behavior for sequences is to become very large as n becomes large. Here are two examples.

$$a_n = 2^n$$

$$a_n = \frac{n^3 + 1}{n^2 + 1}$$

Here is a definition to go along with this idea.

Definition We say that a sequence a_n *diverges to* $+\infty$ if given any positive number M there is an N such that $a_n > M$ whenever $n > N$.

Sequences that neither converge nor diverge

Some sequences have no limit. Here are a couple of examples.

$$a_n = 1 + (-1)^n$$

$$a_n = \tan n$$