Improper integrals

1 An example that leads to an improper integral

Consider a rocket launched from the surface of the earth. How much work is needed to move the rocket from the surface of the earth to a certain height? The force the rocket moves against is the force of gravity,

$$f(x) = -\frac{m\,k}{x^2}$$

where x is the distance from the center of the earth in (say) miles, m is the mass of the rocket, and k is a constant. Let us assume that the rocket has a weight of 100,000 lbs at the surface of the earth, which is roughly 4000 miles from the center of the earth. This allows us to determine $k\,m$.

$$-100000 = \frac{m\,k}{(4000)^2}$$

$$m\,k = 1.6\;10^{12}\;lb\text{ - }mile^2$$

Given all this, how much work does it take to lift the rocket up to, say 400 miles up?

$$W = \int_{4000}^{4400} \frac{m\,k}{x^2}\,dx$$

How much work does it take to lift the rocket up to a point in space where the Earth's gravity is essentially negligible? Certainly if we go all the way out to infinity, the gravitational attraction of the Earth will be negligible.

$$W = \int_{4000}^{\infty} \frac{m\,k}{x^2}\,dx$$

How do we do this integral? The naive way to procede is to pretend that ∞ is a number and do this integral the same way we would do any definite integral.

$$W = -\frac{m\,k}{x}\,\Big|_{4000}^{\infty}$$

The more proper way to do this integral is to side-step the issue of the infinity by means of a limit.

$$W = \lim_{A \to \infty} \left(\int_{4000}^{A} \frac{m\,k}{x^2}\,dx \right) = \lim_{A \to \infty} \left(-\frac{m\,k}{x} \Big|_{4000}^{A} \right)$$

$$= \lim_{A \to \infty} \left(-\frac{m\,k}{A} + \frac{m\,k}{4000} \right) = \frac{m\,k}{4000}$$

The integral with an ∞ in it is an example of an improper integral. The best way to understand improper integrals is to look at a proper integral:

$$\int_a^b f(x)\,dx$$

To make this proper:

1. Both a and b have to be finite numbers.

2. $f(x)$ also has to be continuous over the range $[a,b]$.

How do you deal with an improper integral? Use some sort of limit to put off the problem until you are done with the integral.

Here are some examples:

$$\int_0^{\infty} e^x\,dx = \lim_{A \to \infty} \left(\int_0^A e^x\,dx \right) = \lim_{A \to \infty} \left(e^x \big|_0^A \right) = \lim_{A \to \infty} \left(e^A - 1 \right) = \infty$$

$$\int_0^1 \frac{1}{x^3}\,dx = \lim_{A \to 0} \int_A^1 \frac{1}{x^3}\,dx = \lim_{A \to 0} \left(-1/2\; x^{-2} \right)\big|_A^1 = \lim_{A \to 0} \left(-1/2 + \frac{1}{2\,A^2} \right) = +\infty$$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{A \to 0} \int_A^1 \frac{1}{\sqrt{x}}\,dx = \lim_{A \to 0} \left(2\sqrt{x} \right)\big|_A^1 = \lim_{A \to 0} \left(2 - 2\sqrt{A} \right) = 2$$

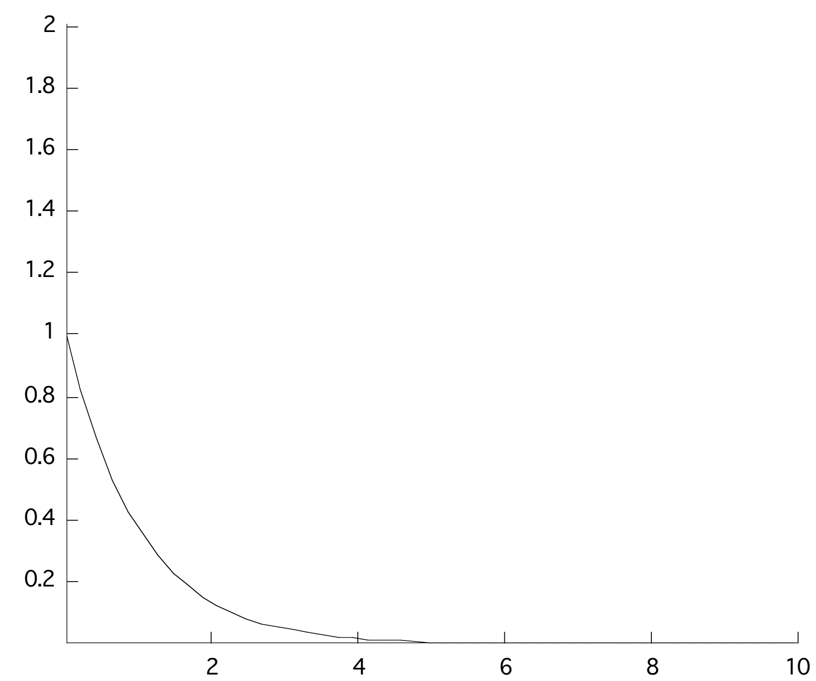
If the limit is finite, we say that the integral *converges*, if it is not finite, we say that the integral *diverges*.

The final example shown here strikes some people as rather odd. The integrand blows up to $+\infty$ as x approaches 0, so you might expect at first that the area under the curve is also infinite. This is actually not the case, because the function does not blow up fast enough to make the area infinite. Note that in the example preceding this one the function does blow up quickly enough to make the area under the curve be infinite.

The next example shows a related thing going on at ∞.

$$\int_0^{\infty} e^{-x}\, dx = \lim_{A \to \infty} \int_0^{A} e^{-x}\, dx = \lim_{A \to \infty} (-e^{-x})|_0^{A} = \lim_{A \to \infty} (-e^{-A} + 1) = 1$$

If you look at a plot of the integrand, you can see that it decays quickly at infinity, thus making it possible for the area under the curve to be finite.



e^{-x}

2 The comparison method

From time to time you will encounter an improper integral that is fairly complicated to compute. If all that you want to know is whether or not the integral is finite, there is a simple comparison technique that can help.

Consider the improper integral

$$\int_1^{\infty} \frac{1}{x^2 + x + 1}\, dx$$

The correct way to handle this integral is to state the problem as a limit problem.

$$\lim_{A \to \infty} \int_1^A \frac{1}{x^2 + x + 1} dx$$

If all that we want to know is whether or not the integral converges, we can compare the integral to a related, simpler integral.

$$\lim_{A \to \infty} \int_1^A \frac{1}{x^2 + x + 1} dx < \lim_{A \to \infty} \int_1^A \frac{1}{x^2} dx$$

The comparison is valid because the inequality

$$\frac{1}{x^2 + x + 1} < \frac{1}{x^2}$$

is valid over the entire range of the integral. The latter integral has the advantage of being easy to do

$$\lim_{A \to \infty} \int_1^A \frac{1}{x^2} dx = \lim_{A \to \infty} (-1/x)\big|_1^A = \lim_{A \to \infty} (-1/A + 1) = 1$$

Since the second integral converges, it forces the original integral to converge as well. This technique is especially useful in cases where the original integral is difficult or impossible to do.

Which integrals converge?

As we have seen, some improper integrals end up being finite, while others fail to converge. Is there an easy way to predict whether or not a given improper integral will converge? The answer to that question is yes: all we have to do is to consider two general examples.

For integrals that are improper because the integral has an infinity as one of its limits, it suffices to consider the example

$$\int_1^\infty x^p dx = \lim_{A \to \infty} \left(\frac{1}{p+1} x^{p+1} \right)\Big|_1^A = \begin{cases} \text{finite} & p < -1 \\ \text{infinite} & p \geq -1 \end{cases}$$

Thus, to judge convergence for this type of integral, we simply have to compare the integrand to x^p. If $p < -1$, we can predict that the integral will converge.

Consider the example

$$\int_1^\infty \frac{\sqrt{x}}{x^2+1}\,dx < \int_1^\infty \frac{\sqrt{x}}{x^2}\,dx = \int_1^\infty x^{-3/2}\,dx$$

Since the exponent is less than -1, the integral all the way on the right will converge. Since the original integral is clearly positive and less than the integral on the right, it too will converge.

For integrals that are improper because the integrand is not finite at some point, it suffices to consider this example:

$$\int_a^b \frac{1}{(x-a)^p}\,dx = \lim_{A \to a^+} \left(\frac{-1}{p-1}\,(x-a)^{-p+1} \Big|_A^b \right) = \begin{cases} \text{finite} & p < 1 \\ \text{infinite} & p \geq 1 \end{cases} \quad (1)$$

Consider this example:

$$\int_0^1 \frac{1}{x^3-1}\,dx$$

The problem here is the fact that the integrand becomes infinite at $x = 1$. At first glance, it might appear that this integral is comparable to

$$\int_{-1}^0 \frac{1}{x^3}\,dx$$

and hence diverges. Well, it turns out ultimately that the integral does diverge, but we have to be a little more careful with our reasoning. The correct way to handle this example is to use partial fractions to isolate the trouble spot.

$$\int_0^1 \frac{1}{x^3-1}\,dx = \int_0^1 \frac{1}{(x-1)(x^2+x+1)}\,dx = \int_0^1 \frac{-x/3-2/3}{x^2+x+1} + \frac{1/3}{x-1}\,dx$$

The first of these two integrals is a proper integral over the interval in question, while the second integral matches the pattern in (1) above with $p = 1$, and hence diverges.

Discontinuity at an interior point

Sometimes the function we are integrating fails to be continuous at some interior point in the interval over which we are integrating. The correct thing to do in that case is to isolate the point of discontinuity and then use limits. Consider this example.

$$\int_0^3 \frac{1}{x^2 - 4}\,dx = \int_0^3 \frac{1/4}{x - 2} - \frac{1/4}{x + 2}\,dx = \int_0^3 \frac{1/4}{x - 2}\,dx - \int_0^3 \frac{1/4}{x + 2}\,dx$$

The integrand in the first integral on the right is discontinuous at $x = 2$, which lies in the interior of the interval [0,3]. We use limits to handle this problem:

$$\lim_{A \to 2^-} \int_0^A \frac{1/4}{x - 2}\,dx + \lim_{A \to 2^+} \int_A^3 \frac{1/4}{x - 2}\,dx$$

$$= \lim_{A \to 2^-} \frac{1}{4} \ln|x - 2| \,\Big|_0^A + \lim_{A \to 2^+} \frac{1}{4} \ln|x - 2| \,\Big|_A^3$$

Both of these limits diverge, so we say that the original integral is undefined.