

Partial Fractions

Introduction to Partial Fractions

Given a rational function of the form

$$\frac{p(x)}{q(x)}$$

where the degree of $p(x)$ is less than the degree of $q(x)$, the method of partial fractions seeks to break this rational function down into the sum of simpler rational functions. In particular, we are going to try to write the original function as a sum of rational functions where the degrees of the polynomials involved are as small as possible.

The steps in the partial fractions method are as follows:

1. Make sure that the degree of the numerator is less than the degree of the denominator. (Below we will see what to do when that is not true.)
2. Factor the denominator as fully as possible.
3. Write the original rational function as a sum of fractions. Each of the terms in the sum gets one of the original factors as its denominator. The numerator of each new fraction is an arbitrary polynomial of degree one less than the degree of the denominator.
4. Use algebra to solve for the coefficients in the numerators of the new fractions.

Here is a step by step tour through the method applied to the rational function

$$\frac{2x + 4}{x^2 - 4x - 5}$$

Note that the degree of the numerator is one less than the degree of the denominator, as required in 1 above.

Next, we factor the polynomial in the denominator.

$$\frac{2x + 4}{x^2 - 4x - 5} = \frac{2x + 4}{(x - 5)(x + 1)}$$

We then attempt to write the fraction as a sum of fractions, with each fraction getting one of the terms in the factored form. The numerator of each fraction is a polynomial with degree one less than that of polynomial in the denominator.

$$\frac{2x + 4}{(x - 5)(x + 1)} = \frac{A}{x - 5} + \frac{B}{x + 1}$$

To solve for the unknown coefficients on the right, we multiply both sides by the denominator of the original rational function.

$$4 + 2x = (-5 + x)(1 + x) \left(\frac{A}{(-5 + x)} + \frac{B}{(1 + x)} \right)$$

We get some nice cancelation on the right, giving us

$$4 + 2x = A(1 + x) + B(x - 5) \tag{1}$$

There are two ways to solve for the unknown coefficients A and B . The simplest way is to make some judicious substitutions for x . If we set $x = -1$, (1) becomes

$$2 = -6B$$

or

$$B = -\frac{1}{3}$$

If we set $x = 5$, (1) becomes

$$14 = 6A$$

or

$$A = \frac{7}{3}$$

An alternative technique is to expand out the expressions on the right and regroup terms to make the right hand side look like a polynomial in x .

$$4 + 2x = (A + B)x + A - 5B$$

Comparing like terms on either side of the equation gives us a set of two equations in two

unknowns.

$$2 = A + B$$

$$4 = A - 5 B$$

I am assuming that you all know how to solve systems of equations, so I will skip to the answer.

$$A = \frac{7}{3}$$

$$B = \frac{-1}{3}$$

By either method, we now can write

$$\frac{2x + 4}{(x - 5)(x + 1)} = \frac{7/3}{x - 5} - \frac{1/3}{x + 1}$$

If the problem is to integrate the original rational function, it is easy to see that it is many times easier to integrate the two simpler rational functions that arise from the partial fraction expansion.

$$\int \frac{2x + 4}{(x - 5)(x + 1)} dx = \int \frac{7/3}{x - 5} dx - \int \frac{1/3}{x + 1} dx$$

The substitution

$$u = x - 5$$

turns the first integral into

$$\int \frac{7}{3} \frac{1}{u} du = \frac{7}{3} \ln u + C$$

Hence

$$\int \frac{7/3}{x - 5} dx = \frac{7}{3} \ln(x - 5) + C$$

Likewise, the substitution

$$u = x + 1$$

turns the second integral into

$$-\int \frac{1}{3} \frac{1}{u} du = -\frac{1}{3} \ln u + C = -\frac{1}{3} \ln(x+1)$$

Putting this all together gives

$$\int \frac{2x+4}{(x-5)(x+1)} dx = \frac{7}{3} \ln(x-5) - \frac{1}{3} \ln(x+1) + C$$

Another example

Suppose we have to integrate

$$\frac{3x+2}{x^3-1}$$

The first step is to factor the denominator as fully as possible.

$$\frac{3x+2}{x^3-1} = \frac{3x+2}{(x-1)(x^2+x+1)}$$

We then write the fraction as a sum of simpler fractions whose denominators are the factors of the original denominator.

$$\frac{3x+2}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

We next multiply both sides by the denominator of the fraction on the left.

$$3x+2 = A(x^2+x+1) + (Bx+C)(x-1) \tag{2}$$

There are two methods for how to proceed next. The first method is to substitute several different possible values for x in an effort to get (2) to reduce to a set of simpler equations.

The first place to look for values of x to substitute is to pick values that cause one or more of the factors on the right to vanish. In this example, the only way to make something vanish on the right is to set $x = 1$. Substituting $x = 1$ gives

$$5 = 3A$$

$$A = 5/3$$

Once we have used up all of the substitutions that make terms vanish, we next try substitutions that are likely to yeild simple results. A popular choice for this stage is $x = 0$. Substituting $x = 0$ gives

$$2 = A - C$$

$$C = -1/3$$

Finally, another simple substitution is $x = -1$:

$$-1 = A - 2(C - B)$$

$$B = -5/3$$

A second method for dealing with equation (2) is to expand the terms on the right hand side and then compare terms with like powers on either side of the equation. This leads to a set of equations

$$\left\{ \begin{array}{l} 0 = A + B \\ 3 = A - B + C \\ 2 = A - C \end{array} \right.$$

We then have to solve that system of equations by some means. I will not show the details, but the result is the same as before:

$$\left\{ \begin{array}{l} A = \frac{5}{3} \\ B = \frac{-5}{3} \\ C = \frac{-1}{3} \end{array} \right.$$

Now that we have determined the coefficients, we can move on to solving the integrals.

$$\int \frac{3x + 2}{x^3 - 1} dx = \int \frac{\frac{5}{3}}{x - 1} dx + \int \frac{-\frac{5}{3}x - \frac{1}{3}}{x^2 + x + 1} dx$$

Both integrals can be handled via a simple substitution. I will leave the details to you.

What to do if the degree of the numerator is too large

The next example shows what to do when the numerator of the rational function has a

degree which is too high.

$$\int \frac{x^3 + x}{x^2 + 2} dx$$

Before we apply partial fractions, we have to apply the polynomial equivalent of long division, which works very much like long division for numbers. Consider this example. What can you do with the fraction

$$\frac{21}{5}$$

to ‘simplify’ it? One thing you can do is to compute the quotient and remainder upon dividing 5 into 21:

$$\frac{21}{5} = 4 + \frac{1}{5}$$

Our method for handling polynomials is similar, and is based on the analog of long division for integers. When confronted with a rational function like

$$\frac{x^3 + x}{x^2 + 2}$$

we will perform long division with polynomials to compute the quotient and remainder.

$$x - \frac{x}{x^2 + 2}$$

Thus

$$\int \frac{x^3 + x}{x^2 + 2} dx = \int x dx - \int \frac{x}{x^2 + 2} dx$$

The first integral on the right is trivial, while the second integral is also not too hard to integrate: we simply apply the substitution $u = x^2 + 2$.

Here then is one last example.

$$\int \frac{x^2 - 2x + 1}{(x + 1)(x^2 + 1)} dx$$

The rational function is in the correct form for partial fractions (the numerator has

degree 2, while the denominator has degree 3), so we go directly to the decomposition.

$$\frac{x^2 - 2x + 1}{(x + 1)(x^2 + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1}$$

Note that the numerators of each of the two new fractions have to have degree one less than their denominators. Next, multiply both sides by the original denominator.

$$x^2 - 2x + 1 = A(x^2 + 1) + (Bx + C)(x + 1)$$

We next make a series of substitutions for x in an effort to produce a set of simpler equations. Substituting $x = -1$ gives

$$4 = 2A$$

$$A = 2$$

Substituting $x = 0$ gives

$$1 = A + C = 2 + C$$

$$C = -1$$

Substituting $x = 1$ gives

$$0 = 2A + 2(B + C) = 4 + 2B - 2$$

$$B = -1$$

Thus

$$\begin{aligned} \int \frac{x^2 - 2x + 1}{(x + 1)(x^2 + 1)} dx &= \int \frac{2}{x + 1} dx + \int \frac{-x - 1}{x^2 + 1} dx = 2 \int \frac{1}{x + 1} dx - \int \frac{x}{x^2 + 1} dx - \\ &\quad \int \frac{1}{x^2 + 1} dx \\ &= 2 \ln|x + 1| - \frac{1}{2} \ln|x^2 + 1| - \tan^{-1} x + C \end{aligned}$$