Trig substitutions

There are number of special forms that suggest a trig substitution. The most common candidates for trig substitutions include the forms

$$\sqrt{a^2 - x^2} \text{ which suggests } x = a \sin\theta \tag{1}$$

$$\sqrt{a^2 + x^2} \text{ which suggests } x = a \tan\theta \tag{2}$$

$$\sqrt{x^2 - a^2} \text{ which suggests } x = a \sec\theta \tag{3}$$

Here are some examples where these substitutions help.

In the first example we compute

$$\int_{-1}^{1} \sqrt{1 - x^2}\, dx$$

This example has an interesting interpretation. What we are computing here is the area of a semicircle of radius 1. We know in advance that the answer should be $\pi/2$.

The recommended substitution in this case is

$$x = \sin\theta$$

$$dx = \cos\theta\ d\theta$$

Applying this substitution gives

$$\int \sqrt{1 - x^2}\, dx = \int \sqrt{1 - \sin^2\theta}\ \cos\theta\, d\theta = \int \cos^2\theta\, d\theta$$

We can solve the cosine squared integral via the substitution

$$\cos^2\theta = \frac{1 + \cos(2\,\theta)}{2}.$$

$$\int \cos^2\theta\, d\theta = \int \frac{1 + \cos(2\,\theta)}{2} d\theta = \frac{1}{2}\,\theta + \frac{1}{4}\sin(2\,\theta) + C$$

The last step is to substitute back for θ by using $\theta = \sin^{-1} x$:

$$\frac{1}{2}\,\theta + \frac{1}{4}\sin(2\,\theta) = \frac{1}{2}\sin^{-1}x + \frac{1}{4}\sin(2\sin^{-1}x) + C$$

Substituting the endpoints and simplifying gives

$$\left(\frac{1}{2}\sin^{-1}1 + \frac{1}{4}\sin(2\sin^{-1}1)\right) - \left(\frac{1}{2}\sin^{-1}(-1) + \frac{1}{4}\sin(2\sin^{-1}(-1))\right) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

This is the expected result.

A more difficult integral

The next example is

$$\int \sqrt{1+x^2}\,dx$$

The suggested substitution is

$$x = \tan\theta$$

which leads to

$$d\,x = \sec^2\theta\;d\,\theta$$

Substituting these back into the integral gives

$$\int \sqrt{1+\tan^2\theta}\,\sec^2\theta\,d\theta$$

$$= \int \sqrt{\sec^2 u}\,\sec^2\theta\,d\theta$$

$$= \int \sec^3\theta\,d\theta$$

You can solve this integral through a clever application of integration by parts. The trick is to rewrite the integral as

$$\int \sec^3\theta\,du = \int \sec^2\theta\,\sec\theta\,d\theta$$

and integrate the $\sec^2\theta$ term while differentiating the $\sec\theta$ term.

$$\int (\sec\theta)^3\,d\theta = \sec\theta\,\tan\theta - \int \sec\theta\,\tan^2\theta\,d\theta$$

We can evaluate the latter integral by a trig identity.

$$\int \sec\theta \tan^2\theta \, d\theta = \int \sec\theta \, (\sec^2\theta - 1) \, d\theta$$

$$= \int -\sec\theta + (\sec\theta)^3 \, d\theta$$

Thus

$$\int (\sec\theta)^3 \, d\theta = \sec\theta \tan\theta + \int \sec\theta \, d\theta - \int (\sec\theta)^3 \, d\theta$$

Rearranging slightly gives

$$2 \int (\sec\theta)^3 \, d\theta = \sec\theta \tan\theta + \int \sec\theta \, d\theta$$

Earlier we determined

$$\int \sec\theta \, d\theta = \ln\left|\sec\theta + \tan\theta\right| + C$$

Hence

$$\int (\sec\theta)^3 \, d\theta = \frac{1}{2}\left(\sec\theta \tan\theta + \ln\left|\sec\theta + \tan\theta\right| + C\right)$$

The last step is to substitute back in for x:

$$x = \tan\theta$$

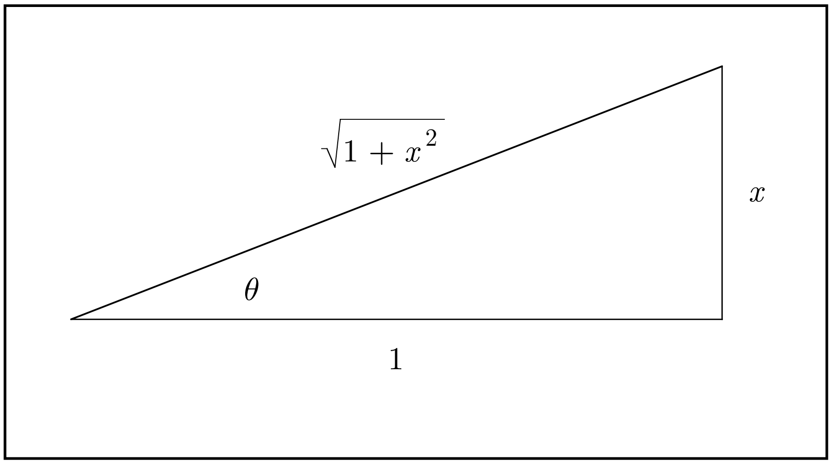
$$\theta = \tan^{-1} x$$

$$\int \sqrt{1 + x^2} \, dx = \frac{1}{2}\left(\sec(\tan^{-1} x) \tan(\tan^{-1} x) + \ln\left|\sec(\tan^{-1} x) + \tan(\tan^{-1} x)\right| + C\right)$$

Our last problem is figuring out how to simplify expressions like $\sec(\tan^{-1} x)$. Instead of substituting back for θ we can try a different approach. Writing the answer in the form

$$\frac{1}{2}\left(\sec\theta \tan\theta + \ln\left|\sec\theta + \tan\theta\right| + C\right)$$

we see that we have to compute $\sec\theta$ given that $\tan\theta = x$. The key to handling situations like this is to go back to the original substitution ($x = \tan\theta$) and interpret it as a statement about a right triangle with angle θ. We can construct a right triangle with an angle whose tangent is x by making the side opposite the angle have length x and the side adjacent to the angle have length 1. This forces the hypotenuse to have length $\sqrt{1 + x^2}$.



We can then read off from this diagram that

$$\tan\theta = x$$

$$\sec\theta = \sqrt{1 + x^2}$$

Thus

$$\int \sqrt{1 + x^2}\, dx = \frac{1}{2}\left(\sec\theta\,\tan\theta + \ln\left|\sec\theta + \tan\theta\right| + C\right) =$$
$$\frac{1}{2}\left(x\,\sqrt{1 + x^2} + \ln\left|\sqrt{1 + x^2} + x\right| + C\right)$$

Another triangle example

The substitutions suggested above really come in useful in integrals in which those square root form appears in combination with other algebraic expressions. Consider this example.

$$\int \frac{\sqrt{4 - x^2}}{x^2}\, dx$$

The suggested substitution in this case is

$$x = 2\,\sin\theta$$

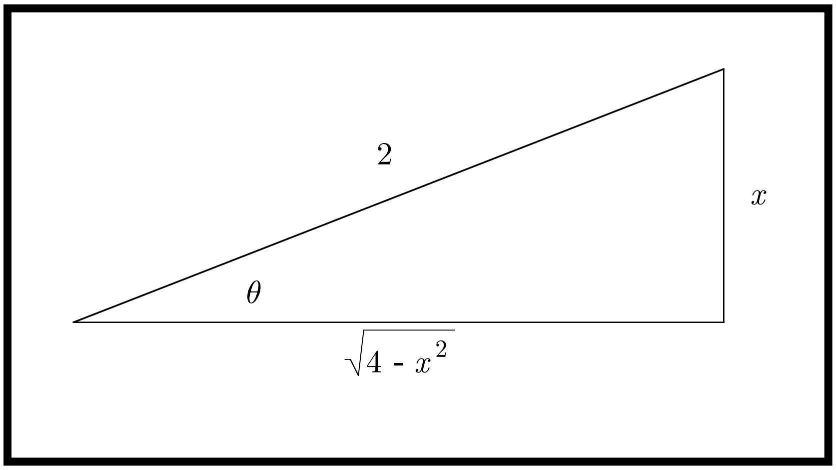
$$d\,x = 2\,\cos\theta\; d\theta$$

Making this substitution converts the integral to

$$\int \frac{\sqrt{4 - 4\sin^2\theta}}{4\sin^2\theta} 2\cos\theta\, d\theta = \int \frac{2\sqrt{1 - \sin^2\theta}}{4\sin^2\theta} 2\cos\theta\, d\theta = \int \frac{\cos^2\theta}{\sin^2\theta} d\theta$$

$$= \int \tan^2\theta\, d\theta = \int \sec^2\theta - 1\, d\theta = \tan\theta - \theta + C$$

Once again the problem at the end is to reverse the substitution. To do this, we can replay the argument we used in the last example with a triangle constructed to ensure that $\sin\theta = x/2$.



We read off from this diagram that $\tan\theta = x/\sqrt{4 - x^2}$:

$$\int \frac{\sqrt{4 - x^2}}{x^2} dx = \frac{x}{\sqrt{4 - x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C$$

Extra algebra work is sometimes needed

The next example shows that sometimes we will have to do some preliminary algebra and a preliminary substitution before we can apply the trig substitution of our choice. Here is the problem:

$$\int \frac{2x}{\sqrt{2x^2 + 3x + 2}} dx$$

The form of the expression in the radical suggests that we should use the substitution appropriate for $x^2 + a^2$, which is $x = a\tan\theta$. However, before we can apply that substitution, we have to make the expression in the radical look more like the form $x^2 + a^2$. The first thing to do is to eliminate the factor of 2 in front of the x^2 term. We can do this by factoring out a factor of 2 from underneath the radical.

$$\int \frac{2\,x}{\sqrt{2\,x^2 + 3\,x + 2}}\,dx = \frac{1}{\sqrt{2}}\int \frac{2\,x}{\sqrt{x^2 + 3/2\,x + 1}}\,dx$$

The next step is to get rid of the superfluous $3/2\ x$ term in the radical expression. The appropriate way to accomplish that is to complete the square in the polynomial.

$$x^2 + 3/2\,x + 1 = x^2 + 2\,(3/4)\,x + 1 = x^2 + 2\,(3/4)\,x + (3/4)^2 - (3/4)^2 + 1$$

$$= (x + 3/4)^2 + 7/16$$

The next step is to introduce a substitution that turns the $(x + 3/4)^2$ term into u^2. The appropriate substitution is

$$u = x + 3/4$$

$$x = u - 3/4$$

$$dx = du$$

With this substitution the original integral becomes

$$\frac{1}{\sqrt{2}}\int \frac{2\,(u - 3/4)}{\sqrt{u^2 + 7/16}}\,dx = \sqrt{2}\int \frac{u}{\sqrt{u^2 + 7/16}}\,du - \frac{3\sqrt{2}}{4}\int \frac{1}{\sqrt{u^2 + 7/16}}\,du$$

Finally, we do the two integrals by two different methods. The first integral can be handled by the substitution

$$w = u^2 + 7/16$$

$$dw = 2\,u\,du$$

With this substitution the first integral becomes

$$\frac{\sqrt{2}}{2}\int w^{-1/2}\,dw = \left(\frac{\sqrt{2}}{2}\right)(2\,w^{1/2}) = \sqrt{2}\,\sqrt{u^2 + 7/16} = \sqrt{2\,x^2 + 3\,x + 2}$$

The second integral requires the use of a trig substitution:

$$u = \frac{\sqrt{7}}{4}\tan\theta$$

$$du = \frac{\sqrt{7}}{4} \sec^2 \theta \; d\theta$$

This converts the second integral into

$$-\frac{3\sqrt{2}}{4} \int \frac{\frac{\sqrt{7}}{4} \sec^2 \theta}{\sqrt{7/16 \tan^2\theta + 7/16}} d\theta = -\frac{3\sqrt{2}}{4} \int \frac{\sec^2 \theta}{\sqrt{\tan^2 \theta + 1}} d\theta = -\frac{3\sqrt{2}}{4} \int \sec\theta \, d\theta$$

$$= -\frac{3\sqrt{2}}{4} \ln\left|\sec\theta + \tan\theta\right| + C$$

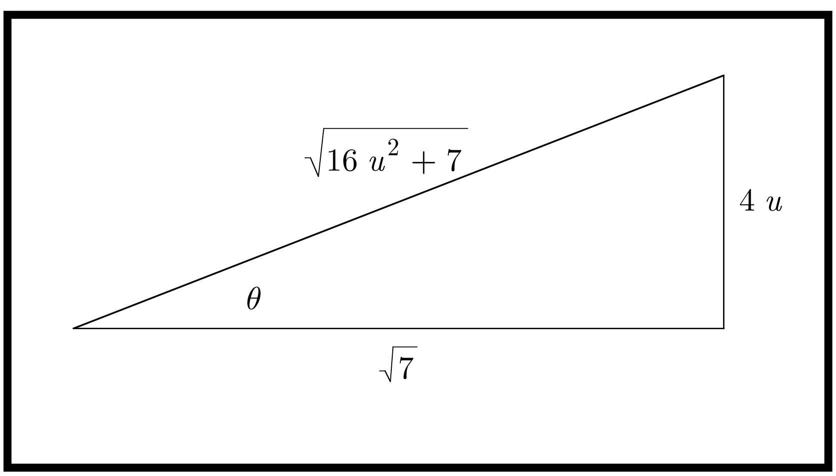
Finally, we have to reverse the trig substitution. The original substitution

$$u = \frac{\sqrt{7}}{4} \tan\theta$$

can be written

$$\frac{4\,u}{\sqrt{7}} = \tan\theta$$

Here is a triangle constructed to make that true.



We can read off from that triangle that $\sec\theta = \sqrt{16\,u^2 + 7}/\sqrt{7} = \sqrt{16/7\,u^2 + 1}$

$$-\frac{3\sqrt{2}}{4} \ln\left|\sec\theta + \tan\theta\right| = -\frac{3\sqrt{2}}{4} \ln\left|\sqrt{16/7\,u^2 + 1} + \frac{4\,u}{\sqrt{7}}\right|$$

$$= - \frac{3\sqrt{2}}{4} \ln\left|\frac{4}{\sqrt{7}} \sqrt{u^2 + 7/16} + \frac{4\,u}{\sqrt{7}}\right| = - \frac{3\sqrt{2}}{4} \ln\left|\frac{4}{\sqrt{7}} \sqrt{2\,x^2 + 3\,x + 2} + \frac{4\,(x + 3/4)}{\sqrt{7}}\right|$$