Trig Integrals

Things we know already

We have already seen how to integrate the sine and cosine functions.

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

Also, in the lecture on integration by parts we saw how to develop reduction formulas for integrals involving certain powers of sine and cosine.

$$\int \sin^n x \, dx = -\frac{1}{n} \cos x \, \sin^{n-1} x + \frac{n-1}{n} \int \sin^{n-2} x \, dx \tag{1}$$

$$\int \cos^n x \, dx = \frac{1}{n} \cos^{n-1} x \, \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx \tag{2}$$

In addition to these facts, there are a number of basic trigonometric identities that appear frequently. The most important of these to know are

$$\sin^2 x + \cos^2 x = 1$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\cos^2 x = 1 - \sin^2 x$$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

A strategy for solving trig integrals

Here is a simple strategy that is useful for solving a wide range of integrals involving trig functions.

1. Convert all trig functions in the integrand into products of sine and cosine.

2. By appropriate use of the substitutions $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$ transform the integral into one of these two forms:

$$\int f(\sin\theta)\cos\theta\,d\theta \text{ or } \int f(\cos\theta)\sin\theta\;d\theta$$

3. Use a substitution $u = \sin\theta$ or $u = \cos\theta$ to solve the integral.

4. If 2 and 3 do not work, try instead turning the integrand into all sine terms or all cosine terms, and then apply reduction formulas (1) or (2).

Some examples

$$\int \sin^3 \mathrm{x} \cos \mathrm{x}\, d\mathrm{x}$$

This example is already in the form shown in step 2 above. A simple substitution

$$u = \sin x$$

$$d\,u = \cos x\;dx$$

converts the original integral to

$$\int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}\sin^4 x + C$$

The next example does require us to use some sort of trig substitution in order to get started.

$$\int \sin^3 x\,\cos^2 x\,dx$$

If we replace two of the sine factors with $(1 - \cos^2 x)$ we will have an expression with all cosine terms and a single sine term.

$$\int \sin^3 x\,\cos^2 x\,dx = \int \sin x\,(1 - \cos^2 x)\cos^2 x\,dx = \int (\cos^2 x - \cos^4 x)\sin x\,dx$$

This integral is very easy to do via the substitution

$$u = \cos x$$

$$du = - \sin x \, dx$$

$$\int (\cos^2 x - \cos^4 x) \sin x \, dx = \int - u^2 + u^4 \, du = -\frac{1}{3} u^3 + \frac{1}{5} u^5 + C$$

Finally, we reverse the substitution by replacing the u terms with $\cos x$.

$$\int \sin^3 x \cos^2 x \, dx = \frac{1}{5} (\cos x)^5 + \frac{1}{3} (\cos x)^3 + C \tag{3}$$

There is almost always an alternative method for solving a trig integration. Another way to procede in this case is to replace the $\cos^2 x$ term with $(1 - \sin^2 x)$.

$$\int \sin^3 x \cos^2 x \, dx = \int \sin^3 x \, (1 - \sin^2 x) \, dx$$

$$= \int \sin^3 x - \sin^5 x \, dx$$

$$= -\int (\sin x)^5 \, dx + \int (\sin x)^3 \, dx \tag{4}$$

We can now apply a reduction formula (1) to solve the two integrals. We start by applying the reduction formula to the power 5 integral:

$$-\int (\sin x)^5 \, dx = \frac{\sin^4 x \cos x}{5} - \frac{4}{5} \int \sin^3 dx$$

This transforms (4) to

$$\frac{\sin^4 x \cos x}{5} - \frac{4}{5} \int \sin^3 dx + \int (\sin x)^3 \, dx = \frac{\sin^4 x \cos x}{5} + \frac{1}{5} \int (\sin x)^3 \, dx$$

Finally, we use the reduction formula one more time to compute the sine cubed integral.

$$\int (\sin x)^3 \, dx = - \frac{\sin^2 x \cos x}{3} + \frac{2}{3} \int \sin x \, dx = - \frac{\sin^2 x \cos x}{3} - \frac{2}{3} \cos x + C$$

Thus

$$\int \sin^3 x \cos^2 x \, dx = -\int (\sin x)^5 \, dx + \int (\sin x)^3 \, dx$$

$$= \frac{\sin^4 x \cos x}{5} + \frac{1}{5} \left(- \frac{\sin^2 x \cos x}{3} - \frac{2}{3} \cos x + C \right)$$

$$= \frac{\sin^4 x \cos x}{5} - \frac{\sin^2 x \cos x}{15} - \frac{2}{15} \cos x + C \tag{5}$$

Note that the two answers (3) and (5), although both correct, do not look very much like each other. This is an unfortunate phenomenon one has to contend with when doing trig integrals. It is not at all unusual to get several correct answers that appear at first glance to be totally different. Because it pretty easy to apply a sequence of trig identities to an expression and transform it into something that looks very different, we should not be surprised to see the same expression rendered in such different ways.

Trigonometric identities that are useful for computing integrals

The following are trig identities that are often useful for transforming trigonometric expressions found in integrands into simpler forms.

$$\sin A \sin B = \frac{1}{2} \cos(A - B) - \frac{1}{2} \cos(A + B) \tag{6}$$

$$\sin A \cos B = \frac{1}{2} \sin(A + B) + \frac{1}{2} \sin(A - B) \tag{7}$$

$$\cos A \cos B = \frac{1}{2} \cos(A - B) + \frac{1}{2} \cos(A + B) \tag{8}$$

$$\sin^2 A = \sin A \sin A = \frac{1}{2} \cos(A - A) - \frac{1}{2} \cos(A + A) = \frac{1 - \cos(2 A)}{2} \tag{9}$$

$$\cos^2 A = \cos A \cos A = \frac{1}{2} \cos(A - A) + \frac{1}{2} \cos(A + A) = \frac{1 + \cos(2 A)}{2} \tag{10}$$

Some examples

The integral

$$\int \sin(2 x) \cos(3 x) \, dx$$

has an integrand that can be replaced via the formula (7) above

$$\int \sin(2 x) \cos(3 x) \, dx = \int \frac{1}{2} \sin(5 x) - \frac{1}{2} \sin(x) \, dx$$

$$= \frac{1}{2} \cos x - \frac{1}{10} \cos(5 x) + C$$

The integral

$$\int \cos^2 \mathrm{x} d \mathrm{x}$$

can be handled via formula (10) above.

$$\int \cos^2 x \, dx = \int \frac{1 + \cos(2 x)}{2} dx = \int \frac{1}{2} dx + \frac{1}{2} \int \cos(2 x) dx = \frac{x}{2} - \frac{1}{4} \sin(2 x) + C$$

Dealing with $\sec x$ **and** $\tan x$

One thing we know about the functions $\sec x$ and $\tan x$ is their derivatives. Those facts can be turned into integration formulas directly.

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

Here are two examples that show how to compute the integrals of $\sec x$ and $\tan x$.

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \int \frac{-1}{u} du = -\ln|u| + C = -\ln|\cos x| + C$$

$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx = \int \frac{\sec^2 x + \sec x \tan x}{\tan x + \sec x} dx$$

$$= \int \frac{1}{u} du = \ln |u| + C = \ln|\sec x + \tan x| + C$$

Other integrals involving combinations of powers of $\sec x$ and $\tan x$ can often be handled by substitutions involving the identities

$$\sec^2 x = 1 + \tan^2 x$$

$$\tan^2 x = \sec^2 x - 1$$

For example,

$$\int \tan^3 x \, dx = \int (\sec^2 x - 1) \tan x \, dx = \int \tan x \sec^2 x \, dx - \int \tan x \, dx$$

$$= \frac{1}{2} (\tan x)^2 - \ln|\sec x| + C$$

Here is another example.

$$\int \tan^3 x \sec x \, dx = \int (\sec^2 x - 1) \sec x \tan x \, dx = \int (\sec x)^2 (\sec x \tan x) \, dx - \int \sec x \tan x \, dx$$

$$= \frac{1}{3}(\sec x)^3 - \sec x + C$$

These two examples demonstrate the general strategy for handling products of powers of $\sec x$ and $\tan x$: if the power of $\sec x$ in the expression is even, try to do substitutions to generate integrals of the form $\int (\tan x)^n \sec^2 x \, dx$ or $\int \tan x \, dx$. In the first case, use the substitution $u = \tan x$ to proceed. If the power of $\sec x$ is odd, use substitutions to generate integrals of the form $\int (\sec x)^n (\sec x \tan x) \, dx$ or $\int \sec x \tan x \, dx$. In the first case, use the substitution $u = \sec x$ to proceed.