

Integration by Parts

Everything we say about antiderivatives comes from some derivative fact. Here is another derivative fact, the product rule:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

If we rearrange the terms and integrate, this leads naturally to the following integral formula

$$\int f'(x)g(x)dx = \int (f(x)g(x))' dx - \int f(x)g'(x)dx$$

Notice the second integral. In this integral we are integrating a derivative - since integration and differentiation are inverses of each other, the integral and the derivative cancel to produce

$$\int f'(x)g(x)dx = f(x)g(x) - \int f(x)g'(x)dx$$

The resulting formula is called the integration by parts formula. Its primary purpose is to transform difficult integrals to easier integrals (when properly applied).

Here is how this can help you in practice. Consider the example

$$\int x \sin x dx$$

The integrand in this example is the product of two terms. We have to decide which of these two terms plays the role of $f'(x)$ and which one plays the role of $g(x)$. The general rule of thumb that guides those choices is that one should pick an $f'(x)$ term that is easy to integrate and a $g(x)$ term with the characteristic that it gets simpler after differentiating. Given these requirements, the following are good choices for this case.

$$f'(x) = \sin x$$

$$f(x) = -\cos x$$

$$g(x) = x$$

$$g'(x) = 1$$

Putting this all together we get

$$\begin{aligned}
\int x \sin x \, dx &= \int f'(x) g(x) \, dx = f(x) g(x) - \int f(x) g'(x) \, dx \\
&= (-\cos x) x - \int -\cos x \, dx \\
&= -x \cos x + \sin x + C
\end{aligned}$$

Here is another example

$$\int x \ln x \, dx$$

Since the log term is hard to integrate, we are essentially forced into integrating the other term instead.

$$f'(x) = x$$

$$f(x) = \frac{1}{2} x^2$$

This is a good choice, because it now leaves us differentiating the log, which in fact produces a simpler result.

$$g(x) = \ln x$$

$$g'(x) = \frac{1}{x}$$

Putting this all together yields

$$\begin{aligned}
\int x^2 \ln x \, dx &= \int f'(x) g(x) \, dx = f(x) g(x) - \int f(x) g'(x) \, dx = \\
&\frac{1}{2} x^2 \ln x - \int \frac{1}{2} x^2 \left(\frac{1}{x}\right) dx = \frac{1}{2} x^2 \ln x - \frac{1}{2} \int x \, dx \\
&= \frac{1}{2} x^2 \ln x - \frac{1}{2} \left(\frac{1}{2} x^2\right) + C
\end{aligned}$$

1 Some more advanced examples of integration by parts

Sometimes it is not immediately obvious that you can apply the integration by parts method. Consider the example

$$\int \arctan x \, dx$$

The problem here is that the integrand does not look like a product. However, a simple

trick will make it a product: just introduce a factor of 1.

$$\int 1 \arctan x \, dx$$

This leads to the following choices

$$f'(x) = 1$$

$$f(x) = x$$

$$g(x) = \arctan x$$

$$g'(x) = \frac{1}{1+x^2}$$

Putting everything together gives

$$\begin{aligned} \int \arctan x \, dx &= f(x) g(x) - \int f(x) g'(x) \, dx \\ &= x \arctan x - \int x \frac{1}{1+x^2} \, dx \end{aligned}$$

We are almost there. The last thing we need to do is to introduce the substitution

$$u = 1 + x^2$$

for the last integral.

$$\frac{du}{dx} = 2x$$

$$\int x \frac{1}{1+x^2} \, dx = \frac{1}{2} \int \frac{1}{1+x^2} 2x \, dx = \frac{1}{2} \int \frac{1}{u} \, du = \frac{1}{2} \ln u = \frac{1}{2} \ln(1+x^2)$$

The final result is

$$\int \arctan x \, dx = x \arctan x - \frac{1}{2} \ln(1+x^2) + C$$

Another famous example similar to the above is the integral

$$\int \ln x \, dx$$

The choices

$$f'(x) = 1$$

$$f(x) = x$$

$$g(x) = \ln x$$

$$g'(x) = \frac{1}{x}$$

make this problem doable:

$$\int \ln x \, dx = \int 1 \ln x \, dx = x \ln x - x \left(\frac{1}{x} \right) = x \ln x - 1$$

Reduction formulas

Many integrals whose integrands look like some function to a power can be solved by some version of the method of integration by parts. Often, this method leads to what are known as reduction formulas. Here is an example.

$$\int (\sin x)^n \, dx$$

In order to apply integration by parts, we have to write the integrand as a product.

$$\int (\sin x)^{n-1} \sin x \, dx$$

Here are the assignments we need to make the integration by parts method work in this case.

$$f'(x) = \sin x$$

$$f(x) = -\cos x$$

$$g(x) = (\sin x)^{n-1}$$

$$g'(x) = (n-1) (\sin x)^{n-2} \cos x$$

Substituting these assignments into the formula

$$\int f'(x) g(x) \, dx = f(x) g(x) - \int f(x) g'(x) \, dx$$

gives

$$\int (\sin x)^{n-1} \sin x dx = -\cos x (\sin x)^{n-1} - \int -\cos x (n-1)(\sin x)^{n-2} \cos x dx \quad (1)$$

A simple transformation changes the integral on the right

$$(n-1) \int \cos^2 x (\sin x)^{n-2} dx$$

to

$$(n-1) \int (1 - \sin^2 x) (\sin x)^{n-2} dx$$

yielding these two integrals

$$(n-1) \int \sin^{n-2} x dx - (n-1) \int \sin^n dx$$

Substituting this into (1) gives

$$\int (\sin x)^{n-1} \sin x dx = -\cos x (\sin x)^{n-1} + (n-1) \int \sin^{n-2} x dx - (n-1) \int \sin^n dx$$

or

$$(n-1) \int \sin^n dx + \int (\sin x)^{n-1} \sin x dx = -\cos x (\sin x)^{n-1} + (n-1) \int \sin^{n-2} x dx$$

$$\int \sin^n x dx = \frac{1}{n} (-\cos x (\sin x)^{n-1} + (n-1) \int \sin^{n-2} x dx)$$

This is not really a complete answer, this is what we call a reduction formula. The idea with a reduction formula is to apply the formula repeatedly, reducing the exponent as we go along, until the exponent is either 1 or 0.

Here is an example.

$$\int \sin^5 x dx = \frac{1}{5} (-\cos x (\sin x)^4 + 4 \int \sin^3 x dx)$$

One application of the reduction formula is not enough, so we apply the reduction formula again.

$$\int \sin^3 x dx = \frac{1}{3} (-\cos x (\sin x)^2 + 2 \int \sin x dx)$$

The last integral we end up with is something we can do by inspection.

$$\int \sin x dx = -\cos x$$

Substitution back in for the integrals we just computed, we get

$$\int \sin^3 x dx = \frac{1}{3} (-\cos x (\sin x)^2 + 2 (-\cos x))$$

$$\int \sin^5 x dx = \frac{1}{5} \left(-\cos x (\sin x)^4 + 4 \left(\frac{1}{3} (-\cos x (\sin x)^2 + 2 (-\cos x)) \right) \right)$$

This simplifies to

$$\int (\sin x)^5 dx = -\frac{8}{15} \cos x - \frac{4}{15} \cos x (\sin x)^2 - \frac{1}{5} \cos x (\sin x)^4$$

This same trick applies to

$$\int \cos^n x dx$$

and any number or other examples.

An alternative notation

Many people use an alternative notation for the integration by parts formula that is more compact.

$$\int u dv = uv - \int v du$$

The idea is still the same: we have an integral whose integrand is a product of two terms. We will end up integrating one of those terms and differentiating the other.

Here is one last example of integration by parts. This particular example is interesting because it demonstrates that sometimes integration by parts is useful even when it appears to not be working.

$$\int e^x \cos x dx$$

To apply integration by parts in this case, we make assignments

$$u = e^x$$

$$du = e^x$$

$$dv = \cos x$$

$$v = \sin x$$

Putting these assignments into the integration by parts formula

$$\int u dv = u v - \int v du$$

gives

$$\int e^x \cos x dx = e^x \sin x - \int \sin x e^x dx \quad (2)$$

To handle the integral on the right

$$\int \sin x e^x dx$$

we apply integration by parts again with

$$u = e^x$$

$$d u = e^x$$

$$d v = \sin x$$

$$v = -\cos x$$

which gives us

$$\int \sin x e^x dx = e^x (-\cos x) - \int (-\cos x) e^x dx \quad (3)$$

Substituting (3) back into (2) gives

$$\int e^x \cos x dx = e^x \sin x + e^x (\cos x) - \int (\cos x) e^x dx$$

At first glance, we appear to have run in circles: we started with an integral and ended up transforming into the very same integral. Fortunately, some simple algebra will save the day. Adding the integral to both sides gives

$$2 \int \cos x e^x dx = \cos x e^x + e^x \sin x$$

or

$$\int \cos x e^x dx = \frac{1}{2} (\cos x e^x + e^x \sin x) + C$$

which solves the problem.

Definite integrals and integration by parts

Up to this point, we have only used integration by parts to solve indefinite integrals (that is, integrals without limits). The method works just as well for definite integrations. Here is the formula in that case.

$$\int_a^b f'(x) g(x) dx = f(x) g(x)|_a^b - \int_a^b f(x) g'(x) dx$$

Here is an example of an integration by parts problem with limits.

$$\int_1^2 x^2 e^{-x} dx$$

The best way to proceed is to ignore the limits until we get to the very end of the problem. The integral

$$\int x^2 e^{-x} dx$$

suggests integration by parts with the assignments

$$u = x^2$$

$$d u = 2 x$$

$$d v = e^{-x}$$

$$v = - e^{-x}$$

These assignments produce

$$\int x^2 e^{-x} dx = x^2 (- e^{-x}) + \int e^{-x} 2 x dx \tag{4}$$

The integral on the right

$$2 \int e^{-x} x dx$$

can be computed via integration by parts with assignments

$$u = x$$

$$d u = 1$$

$$d v = e^{-x}$$

$$v = - e^{-x}$$

giving

$$2 \int e^{-x} x dx = 2(x (-e^{-x}) + \int e^{-x} dx) \quad (5)$$

The final integral that appears is simple enough to do by hand.

$$\int e^{-x} dx = -\frac{1}{e^x} + C \quad (6)$$

Substituting (6) back into (5) gives

$$2 \int e^{-x} x dx = 2(x (-e^{-x}) + -e^{-x})$$

Substituting this result in turn back into (4) gives

$$\int x^2 e^{-x} dx = x^2 (-e^{-x}) + 2(x (-e^{-x}) + -e^{-x})$$

Now that we have solved the problem without limits completely, we can re-introduce the limits

$$\begin{aligned} \int_1^2 x^2 e^{-x} dx &= (x^2 (-e^{-x}) + 2(x (-e^{-x}) + -e^{-x}))|_1^2 \\ &= (-10 e^{-2}) - (-5 e^{-1}) \end{aligned}$$