

Basic Integration Formulas and the Substitution Rule

1 The second fundamental theorem of integral calculus

Recall from the last lecture the second fundamental theorem of integral calculus.

Theorem Let $f(x)$ be a continuous function on the interval $[a,b]$. Let $F(x)$ be any function with the property that

$$F'(x) = f(x)$$

Then

$$\int_a^b f(x) dx = F(b) - F(a)$$

2 Antiderivatives

Let us now see how the second fundamental theorem gets applied to solve area problems.

Example Compute

$$\int_{10}^{20} \frac{1}{x^2} dx$$

According to the theorem, we need to find a function $F(x)$ with the property

$$F'(x) = 1/x^2$$

Such a function is called an anti-derivative of $1/x^2 = x^{-2}$. We can see that one possible solution is

$$F(x) = -x^{-1}$$

From the theorem

$$\int_{10}^{20} \frac{1}{x^2} dx = F(20) - F(10) = (-(20)^{-1}) - (-(10)^{-1}) = \frac{1}{20}$$

Unfortunately, this anti-differentiation process is not always easy to do.

$$\int_0^1 \sqrt{1-x^2} dx$$

To solve this, we need an anti-derivative

$$F'(x) = \sqrt{1-x^2}$$

There is no obvious anti-derivative for this function. Given the fact that many anti-derivative problems are hard to do, we will be spending a good chunk of this course working out techniques for computing anti-derivatives.

Antiderivative notation

Since we are going to be computing antiderivatives, one of the first things we are going to need is a convenient notation for antidifferentiation. The notation that is commonly used for the operation of antidifferentiation is called the *indefinite integral*.

$$\int f(x) dx = F(x) \Leftrightarrow F'(x) = f(x)$$

Antiderivatives we can compute

One way to get a start on the problem of computing antiderivatives is to write down the derivatives of some standard functions and then adjust those formulas to make them antidifferentiation formulas.

$f(x)$	$F(x) = \int f(x) dx$
x^n	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln x $
e^x	e^x
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\frac{1}{1+x^2}$	$\tan^{-1} x$

(There is a more extensive list of anti-differentiation formulas on page 406 of the text.) Another useful tool for computing antiderivatives is related to the fact that the derivative of a sum is a sum of derivatives:

$$\int f(x) + g(x) dx = \int f(x) dx + \int g(x) dx$$

Likewise, since the derivative of $k f(x)$ is $k f'(x)$ we have

$$\int k f(x) dx = k \int f(x) dx$$

Some Examples

$$\begin{aligned} \int \frac{5 - 4x^3 + 2x^6}{x^6} dx &= \int 5x^{-6} - 4x^{-3} + 2 dx \\ &= 5 \int x^{-6} dx - 4 \int x^{-3} dx + 2 \int 1 dx \\ &= 5 \left(\frac{x^{-5}}{-5} \right) - 4 \left(\frac{x^{-2}}{-2} \right) + 2(x) \\ &= -x^{-5} + 2x^{-2} + 2x \end{aligned}$$

$$\begin{aligned} \int \frac{x^2}{x^2 + 1} dx &= \int \frac{x^2 + 1 - 1}{x^2 + 1} dx = \int 1 - \frac{1}{1 + x^2} dx \\ &= x - \tan^{-1} x \end{aligned}$$

$$\int \frac{1}{2x + 3} dx = \frac{1}{2} \ln(2x + 3)$$

If you want to check whether or not the answer to an antidifferentiation problem is correct, you can always differentiate the answer to see whether or not that returns the original function.

$$\frac{d}{dx} \left(\frac{1}{2} \ln(2x + 3) \right) = \frac{1}{2} \frac{d}{dx} (\ln(2x + 3)) = \frac{1}{2} \frac{2}{2x + 3} = \frac{1}{2x + 3}$$

3 A method based on the chain rule

Since integration is the inverse of differentiation, many differentiation rules lead to corresponding integration rules. Consider, for example, the chain rule.

$$\frac{d}{dx} f(g(x)) = f'(g(x)) g'(x)$$

The chain rule says that when we take the derivative of one function composed with another the result is the derivative of the outer function times the derivative of the inner function. How does this lead to an integration formula? Well, suppose you can figure out

the original $f(x)$ from $f'(x)$ by integrating:

$$\frac{d}{dx} f(x) = f'(x)$$

What happens when you try to differentiate $f(g(x))$?

$$(f(g(x)))' = f'(g(x)) g'(x) \tag{1}$$

What does that tell us? Integrating both sides of (1) gives

$$\int f'(g(x)) g'(x) dx = f(g(x))$$

The formula forms the basis for a method of integration called the substitution method.

Some simple examples

Here are some simple examples where you can apply this technique.

Consider the integral

$$\int (2x + 1)^3 dx$$

To view the thing we are integrating as a composition, introduce the functions

$$f'(x) = x^3, g(x) = 2x + 1$$

This does the job in the sense that

$$f'(g(x)) = (2x + 1)^3$$

but recall that the pattern we have to try and force the problem into is

$$\int f'(g(x)) g'(x) dx$$

What we need to provide is the missing $g'(x)$ term. From what we said above we have

$$g'(x) = 2$$

Introducing a factor of 2 into the problem is easy - we just have to balance it with a corresponding factor of 1/2:

$$\frac{1}{2} \int (2x + 1)^3 dx$$

This now matches the desired form. The next thing we have to do is to recover the original $f(x)$ from $f'(x)$ by finding an anti-derivative.

$$\frac{d}{dx} f(x) = f'(x) = x^3$$

$$f(x) = \frac{1}{4} x^4$$

Putting all of this together with the formula derived above

$$\int f'(g(x)) g'(x) dx = f(g(x))$$

gives

$$\frac{1}{2} \int (2x + 1)^3 dx = \frac{1}{2} \left(\frac{1}{4} (2x + 1)^4 \right)$$

For the next example consider

$$\int \frac{1}{1+x} dx$$

Although it is a little more difficult to come up with at first, there is a composition here:

$$f'(x) = 1/x, g(x) = 1 + x$$

Since $g'(x) = 1$, we don't have to manufacture the missing $g'(x)$ term

$$f'(g(x)) g'(x) = \frac{1}{1+x} \cdot 1$$

We also need the anti-derivative of $f'(x)$

$$\int f'(x) dx = \int 1/x dx = \ln|x| = f(x)$$

$$\int f'(g(x)) g'(x) dx = f(g(x))$$

$$\int \frac{1}{1+x} dx = \ln|x+1|$$

Sometimes finding the right composition takes a little extra work. Consider this problem.

$$\int \frac{x}{1+x^4} dx$$

Recall that we have to match the integrand to the form

$$f'(g(x)) g'(x)$$

It is a little hard to see, but the right thing to do is to set

$$g(x) = x^2$$

$$g'(x) = 2x$$

This shows us how to get the $g'(x)$ term that we need.

$$\frac{1}{2} \int \frac{1}{1+x^4} (2x) dx$$

What remains is to figure out how to write the other term as a composition $f'(g(x))$:

$$\frac{1}{1+x^4} = \frac{1}{1+(x^2)^2}$$

So we see that

$$f'(x) = \frac{1}{1+x^2}$$

$$f'(g(x)) = \frac{1}{1+(x^2)^2} = \frac{1}{1+x^4}$$

We will need to get the anti-derivative of $f'(x)$:

$$\int \frac{1}{1+x^2} dx = \arctan x + c$$

$$f(x) = \arctan x$$

Putting this all together gives us

$$\int f'(g(x)) g'(x) dx = f(g(x))$$

$$\frac{1}{2} \int \frac{1}{1+x^4} (2x) dx = \frac{1}{2} \arctan(x^2)$$

Summary of the method

1. Look for something in the integrand that looks like the composition of two functions, $f'(g(x))$.
2. Find something in the integrand that looks like the derivative of the inner function, $g'(x)$. If necessary, throw in constant factors needed to create the $g'(x)$ term.
3. Find the antiderivative of $f'(x)$.
4. The antiderivative of $f'(g(x)) g'(x)$ is $f(g(x))$.

4 Commonly used notation for substitution

I used f and g above to make clear the origins of this newest method in the chain rule. In practice, people tend to use slightly different notation which suggests a slightly different way to understand what is going on here. Here is an example of that method in use.

Consider the integral

$$\int e^{3x} dx$$

The form of the integrand suggest that we try to transform the problem into something like

$$\int e^u du$$

The substitution that gets this to happen is

$$u = 3x$$

We need a derivative in order to apply this method, so we compute

$$\frac{du}{dx} = 3$$

With these substitutions, the original integral looks like

$$\int e^{3x} dx = \frac{1}{3} \int e^u 3 dx = \frac{1}{3} \int e^u \frac{du}{dx} dx$$

Can we turn this integral in x into an integral in u ? The answer is yes: the key requirement is that we have the $\frac{du}{dx} dx$ combination. By the chain rule, those two pieces can be combined into a single du , completing the transition to an integral in u instead of x .

$$\int f(u) \frac{du}{dx} dx = \int f(u) du$$

At this point, we complete the change of variables:

$$\int e^{3x} dx = \frac{1}{3} \int e^u 3 dx = \frac{1}{3} \int e^u du = \frac{1}{3} e^u + C = \frac{1}{3} e^{3x}$$

5 Substitution and Definite Integrals

We have seen that an appropriately chosen substitution can make an anti-differentiation problem doable. Something to watch for is the interaction between substitution and definite integrals. Consider the following example.

$$\int_{-1}^1 x \sqrt{1-x^2} dx$$

There are two approaches we can take in solving this problem:

1. Use substitution to compute the antiderivative and then use the anti-derivative to solve the definite integral.

$$u = 1 - x^2$$

$$\frac{du}{dx} = -2x$$

$$\begin{aligned} \int x \sqrt{1-x^2} dx &= -\frac{1}{2} \int \sqrt{1-x^2} (-2x) dx \\ &= -\frac{1}{2} \int \sqrt{u} du = -\frac{1}{2} \int u^{1/2} du = -\frac{1}{2} \left(\frac{2}{3} u^{3/2} \right) + C \\ &= -\frac{1}{3} (1-x^2)^{3/2} + C \end{aligned}$$

$$\int_{-1}^1 x \sqrt{1-x^2} dx = -\frac{1}{3} (1-(1)^2)^{3/2} - \left(-\frac{1}{3} (1-(-1)^2)^{3/2} \right) = 0$$

2. Use substitution on both the expression being integrated and on the limits of the integral.

$$\int_{-1}^1 x \sqrt{1-x^2} dx = -\frac{1}{2} \int_{-1}^1 \sqrt{1-x^2} (-2x) dx = -\frac{1}{2} \int_{1-(1)^2}^{1-(-1)^2} \sqrt{u} du = -\frac{1}{2} \int_0^0 \sqrt{u} du$$

In the second approach we can see that the integral will be 0 even without computing an anti-derivative.

6 Antiderivatives may differ by a constant

When we compute an antiderivative, the answer we get can be modified by adding a constant term and still be a valid antiderivative. Here is a simple example. The antiderivative of

$$f(x) = 2x$$

is

$$F(x) = x^2$$

An equally valid antiderivative is

$$G(x) = x^2 + 1$$

One way to capture this flexibility is to say that all of our antiderivatives are valid up to some constant factor by saying something like

$$\int 2 x dx = x^2 + C$$

A common consequence of this effect is that doing the same problem in two different ways can produce two answers that appear to be very different. As long as we can show that the two answers differ by a constant, both answers are legitimate. This phenomenon is most pronounced when working with trig functions.

Here is an example to demonstrate this effect. Consider the integral

$$\int \sin(2 x) dx$$

The form of the integrand suggests that we transform the problem into something like

$$\int \sin u du = -\cos u + C$$

The substitution we need to do this is

$$u = 2 x$$

$$\frac{du}{dx} = 2$$

To get the derivative term we need we will have to introduce a factor of 1/2 to balance the 2.

$$\frac{1}{2} \int \sin(2 x) 2 dx = \frac{1}{2} \int \sin u \frac{du}{dx} dx = \frac{1}{2} \int \sin u du = -\frac{1}{2} \cos(2 x) + C$$

Here is a different approach. We begin by using a trig identity to change the form of the problem.

$$\int \sin(2 x) dx = \int 2 \sin x \cos x dx$$

The latter form can be handled by composition by using

$$g(x) = \sin x$$

$$g'(x) = \cos x$$

$$f'(x) = 2 x$$

$$f(x) = x^2$$

This leads to

$$\int 2 \sin x \cos x dx = \int f'(g(x)) g'(x) dx = f(g(x)) + C = (\sin x)^2 + C$$

This appears to be very different from the earlier answer. However, some manipulation with trig identities will bring the two forms closer together.

$$-\frac{1}{2} \cos(2x) = -\frac{1}{2} (\cos^2 x - \sin^2 x) = -\frac{1}{2} ((1 - \sin^2 x) - \sin^2 x) = -\frac{1}{2} + \sin^2 x$$

The difference between the two solutions is now just $-1/2$, which is acceptable.

7 Homework Problems

Section 5.4: 6, 8, 10, 26, 36

Section 5.5: 12, 32, 44, 58, 64