The Heat Equation

The heat equation is a PDE that models the flow of heat in a long, thin metal bar running from $x = 0$ to $x = l$. $u(x,t)$ is the temperature of the bar at location x at time t.

$$\rho\ c\ \frac{\partial u}{\partial t} - \kappa\ \frac{\partial^2 u}{\partial x^2} = f(x,t)$$

Here ρ is the density of the bar, measured in grams per cubic centimeter, c is the heat capacity of the material, measured in Joules per (gram * degree Kelvin), κ is the thermal conductivity of the material measured in Joules per (centimeter * second * degree Kelvin).

The function $f(x,t)$ models external heat input to the system. In the homogeneous form of the heat equation this external input is 0.

The full problem specifies both a PDE to solve and appropriate boundary conditions. In this case, appropriate boundary conditions include an initial temperature distribution $\psi(x)$ for the bar:

$$\rho\ c\ \frac{\partial u}{\partial t} - \kappa\ \frac{\partial^2 u}{\partial x^2} = f(x,t)$$

$$u(0,t) = u(l,t) = 0$$

$$u(x,0) = \psi(x)$$

This form of the problem uses Dirichlet boundary conditions $u(0,t) = u(l,t) = 0$ which say that the ends of the bar are kept at a constant temperature of 0 throughout.

Solving by the Method of Fourier Series

Using a strategy similar to that we used in the Galerkin method, we can seek to solve this problem by multiplying both sides of the equation by a test function that satisfies the same Dirichlet boundary conditions in x that the solution u(x,t) is supposed to satisfy and then integrating over the interval $[0,l]$. The appropriate test functions to use in this case are the functions $\sin(n\ \pi\ x/l)$:

$$\frac{2}{l}\int_0^l \left(\rho\ c\ \frac{\partial u}{\partial t} - \kappa\ \frac{\partial^2 u}{\partial x^2}\right) \sin\left(\frac{n\ \pi\ x}{l}\right) dx = \frac{2}{l}\int_0^l f(x,t)\ \sin\left(\frac{n\ \pi\ x}{l}\right) dx$$

On the left we recognize two terms:

$$\frac{2}{l}\int_0^l \rho\ c\ \frac{\partial u}{\partial t} \sin\left(\frac{n\ \pi\ x}{l}\right) dx = \rho\ c\ \frac{\partial}{\partial t}\left(\frac{2}{l}\int u(x,t)\ \sin\left(\frac{n\ \pi\ x}{l}\right) dx\right)$$

$$\frac{2}{l}\int_0^l \left(- \kappa\ \frac{\partial^2 u}{\partial x^2}\right) \sin\left(\frac{n\ \pi\ x}{l}\right) dx = - \kappa\ \left(\frac{2}{l}\ \frac{\partial u}{\partial x}\ \sin\left(\frac{n\ \pi\ x}{l}\right)\right)\Big|_0^l + \kappa\ \frac{2\ n\ \pi}{l^2}\ \int_0^l \frac{\partial u}{\partial x}\ \cos\left(\frac{n\ \pi\ x}{l}\right) dx$$

$$= \kappa\ \frac{2\ n\ \pi}{l^2}\ u(x,t)\ \cos\left(\frac{n\ \pi\ x}{l}\right)\Big|_0^l + \kappa\ \frac{2\ n^2\ \pi^2}{l^3}\ \int_0^l u(x,t)\ \sin\left(\frac{n\ \pi\ x}{l}\right) dx$$

$$= \kappa \frac{n^2 \pi^2}{l^2} \frac{2}{l} \int_0^l u(x,t) \sin\left(\frac{n \pi x}{l}\right) dx$$

All of the terms that are left after these manipulations look like Fourier sine transforms.

$$\frac{2}{l}\int_0^l f(x,t) \sin\left(\frac{n \pi x}{l}\right) dx = c_n(t)$$

is the n^{th} Fourier sine coefficient of $f(x,t)$, while

$$\frac{2}{l} \int_0^l u(x,t) \sin\left(\frac{n \pi x}{l}\right) dx = a_n(t)$$

is the n^{th} Fourier sine coefficient of $u(x,t)$.

Putting this all together we have

$$\rho\ c \frac{\partial}{\partial t}\left(\frac{2}{l}\int u(x,t) \sin\left(\frac{n \pi x}{l}\right) dx\right) + \kappa \frac{n^2 \pi^2}{l^2} \frac{2}{l} \int_0^l u(x,t) \sin\left(\frac{n \pi x}{l}\right) dx = \frac{2}{l}\int_0^l f(x,t) \sin\left(\frac{n \pi x}{l}\right) dx$$

$$\rho\ c \frac{\mathrm{d}a_n(t)}{\mathrm{d}t} + \kappa \frac{n^2 \pi^2}{l^2} a_n(t) = c_n(t)$$

which gives us a set of first order ODEs that have to be satisfied by the unknown coefficients $a_n(t)$. These ODEs all need initial conditions, which we can obtain from the initial conditions of the PDE above:

$$u(x,0) = \psi(x)$$

$$a_n(0) = \frac{2}{l} \int_0^l u(x,0) \sin\left(\frac{n \pi x}{l}\right) dx = \frac{2}{l} \int_0^l \psi(x) \sin\left(\frac{n \pi x}{l}\right) dx = b_n$$

The resulting ODEs

$$\rho\ c \frac{\mathrm{d}a_n(t)}{\mathrm{d}t} + \kappa \frac{n^2 \pi^2}{l^2} a_n(t) = c_n(t)$$

$$a_n(0) = b_n$$

can be solved by standard methods from Math 210.

$$a_n(t) = b_n\, e^{-\frac{\kappa\, n^2 \pi^2}{\rho\, c\, l^2} t} + \frac{1}{\rho\ c} \int_0^t e^{-\frac{\kappa\, n^2 \pi^2}{\rho\, c\, l^2} (t-s)}\, c_n(s)\, ds$$

The solution then is expressed as a Fourier sine series with these coefficients.

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n \pi x}{l}\right)$$

Neumann Boundary Conditions

Another physically realistic and important variant of the heat equation uses Neumann boundary conditions at the end of the rod, which say that heat does not flow past either end of the rod:

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(l,t) = 0$$

In this version of the problem we will need test functions that satisfy these boundary conditions. The appropriate functions to use in this case are

$$\cos\left(\frac{n\,\pi\,x}{l}\right)$$

where n ranges from 0 to ∞.

Following reasoning similar to what we used above we get that

$$\frac{2}{l}\int_0^l f(x,t)\,\cos\left(\frac{n\,\pi\,x}{l}\right)\,dx = c_n(t)$$

$$\frac{2}{l}\int_0^l \psi(x)\,\cos\left(\frac{n\,\pi\,x}{l}\right)\,dx = b_n$$

$$u(x,t) = \sum_{n=0}^{\infty} a_n(t)\,\cos\left(\frac{n\,\pi\,x}{l}\right)$$

$$\rho\,c\,\frac{\mathrm{d}a_n(t)}{\mathrm{d}t} + \kappa\,\frac{n^2\,\pi^2}{l^2}\,a_n(t) = c_n(t)$$

$$a_n(t) = b_n\,e^{-\frac{\kappa\,n^2\,\pi^2}{\rho\,c\,l^2}\,t} + \frac{1}{\rho\,c}\int_0^t e^{-\frac{\kappa\,n^2\,\pi^2}{\rho\,c\,l^2}\,(t-s)}\,c_n(s)\,ds$$