Sneaking up on harder problems

Most of the material in an introductory course in partial differential equations deals with a small set of relatively simple PDEs, such as the basic heat equation, wave equation, and Laplace equation. Once one moves beyond those elementary equations, things quickly get uglier. For example, we have seen may different ways to deal with the heat equation

$$\rho\ c\ \frac{\partial u(x,t)}{\partial t} - \kappa\ \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t)$$

This equation is relatively easy to deal with if one assumes uniform material properties. A somewhat more difficult case is the case of non-uniform material properties. In this case the various physical constants ρ, c, and κ are not constants, but may vary at different points in the material. The equation for this case

$$\rho(x)\ c(x)\ \frac{\partial u(x,t)}{\partial t} - \kappa(x)\ \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t)$$

can not be solved by any of the elementary methods we have seen so far.

This is not to say that all of the ideas we have encountered up to this point are completely worthless. In fact, we can attack this problem by using a strategy that attempts to replicate a spectral method such as a Fourier series method.

Here's how this might work in practice.

1. Rewrite the equation slightly to simplify the t derivative term(s).

$$\frac{\partial u(x,t)}{\partial t} - \frac{\kappa(x)}{\rho(x)\ c(x)}\ \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{f(x,t)}{\rho(x)\ c(x)}$$

$$\frac{\partial u(x,t)}{\partial t} - k(x)\ \frac{\partial^2 u(x,t)}{\partial x^2} = g(x,t)$$

2. Find eigenvalues λ_n and eigenfunctions $\varphi_n(x)$ of the time-independent steady-state problem with appropriate boundary conditions.

$$- k(x)\ \frac{\mathrm{d}^2 \varphi_n(x)}{\mathrm{d}x^2} = \lambda_n\ \varphi_n(x)$$

$$- \frac{\mathrm{d}^2 \varphi_n(x)}{\mathrm{d}x^2} = \lambda_n\ w(x)\ \varphi_n(x)$$

3. Express the solution we seek as a combination of these eigenfunctions, and express the forcing function as a combination of the eigenfunctions.

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\ \varphi_n(x)$$

$$g(x,t) = \sum_{n=1}^{\infty} c_n(t)\ \varphi_n(x)$$

4. Substitute these into the PDE to generate a family of ODEs to solve for the coefficients $a_n(t)$.

$$a_n'(t)\,\varphi_n(x) + \lambda_n\, a_n(t)\,\varphi_n(x) = c_n(t)\,\varphi_n(x)$$

5. Solve the ODEs for the coefficients $a_n(t)$ and construct the solution

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\varphi_n(x)$$

Sturm - Liouville boundary value problems

An essential step in the plan laid out above is the step where we find eigenvalues and eigenfunctions of the steady-state solution by solving a BVP. To generalize the particular BVP we saw above slightly, we will consider a family of BVPs called the Sturm-Liouville BVPs.

$$-\frac{d}{dx}\left(P(x)\,\frac{du(x)}{dx}\right) + R(x)\,u(x) = \lambda\,w(x)\,u(x)$$

$$\alpha_1\,u(a) + \alpha_2\,\frac{du}{dx}(a) = 0$$

$$\beta_1\,u(b) + \beta_2\,\frac{du}{dx}(b) = 0$$

Here it is assumed that the functions $P(x)$ and $w(x)$ are both positive on the interval $[a,b]$.

The first practical problem we will encounter when working with Sturm-Liouville BVPs is that many of these problems can not be solved by elementary ODE techniques. The textbook shows one simple example

$$-\frac{d}{dx}\left(x\,\frac{du(x)}{dx}\right) = \lambda\,\frac{1}{x}\,u(x)$$

which can solved by rewriting the equation as

$$x^2\,\frac{d^2u(x)}{dx^2} + x\,\frac{du(x)}{dx} + \lambda\,u(x) = 0$$

and noticing that this equation is an Euler equation and solving it by the usual method for solving an Euler equation.

Beyond a few, simple examples, the only reasonable approach is to use a tool like Mathematica to solve the BVP, apply a more advanced ODE solving technique, or use a numerical approximation method to solve the BVP. The accompanying Mathematica notebook shows some examples of more difficult BVPs, while the next lecture will demonstrate how to use the finite element method to construct approximate solutions to Sturm-Liouville equations.

Some theoretical results

The Sturm-Liouville BVP

$$-\frac{d}{dx}\left(P(x)\,\frac{du(x)}{dx}\right) + R(x)\,u(x) = \lambda\,w(x)\,u(x)$$

$$\alpha_1\, u(a) + \alpha_2\, \frac{\mathrm{d}u}{\mathrm{d}x}(a) = 0$$

$$\beta_1\, u(b) + \beta_2\, \frac{\mathrm{d}u}{\mathrm{d}x}(b) = 0$$

is essentially the problem of finding eigenvalues and eigenfunctions of a differential operator

$$L\, u = \frac{1}{w(x)}\left(- \frac{\mathrm{d}}{\mathrm{d}x}\left(P(x)\, \frac{\mathrm{d}u(x)}{\mathrm{d}x}\right) + R(x)\, u(x)\right)$$

with appropriate boundary conditions. The first thing to say about this operator is that in general it will not be symmetric with respect to the standard inner product

$$(f,g) = \int_a^b f(x)\, g(x)\, \mathrm{d}\, x$$

It is possible to show, however, that this operator is symmetric with respect to a modified inner product

$$(f,g)_w = \int_a^b f(x)\, g(x)\, w(x)\, \mathrm{d}\, x$$

Here now are some theoretical results about the nature of eigenvalues and eigenfunctions of the linear operator L. As the textbook points out, the proofs of most of these results are beyond the scope of this course, so we present this list of results without proof.

1. L has an infinite sequence of real eigenvalues $\lambda_1 < \lambda_2 < \cdots$, with $\lambda_n \to \infty$ as $n \to \infty$.

2. Associated with each eigenvalue λ_n is a single eigenfunction $\varphi_n(x)$.

3. Eigenfunctions $\varphi_n(x)$ corresponding to distinct eigenvalues λ_n are orthogonal with respect to the $(\ ,\ )_w$ inner product.

4. Assuming that the eigenfunctions $\varphi_n(x)$ have been normalized with respect to the $(\ ,\ )_w$ inner product so that $(\varphi_n(x),\varphi_n(x))_w = 1$, we can write functions defined on $[a,b]$ as combinations of the eigenfunctions $\varphi_n(x)$:

$$f(x) = \sum_{n=1}^{\infty} a_n\, \varphi_n(x)$$

$$a_n = (f(x),\varphi_n(x))_w = \int_a^b f(x)\, \varphi_n(x)\, w(x)\, \mathrm{d}\, x$$

5. The eigenfunction $\varphi_n(x)$ has exactly n - 1 zeros in the open interval (a,b) and each zero of $\varphi_n(x)$ lies between two consecutive zeros of $\varphi_{n+1}(x)$.

Solving Sturm-Liouville Boundary Value Problems

The text treats the ODEs that arise when we solve Sturm-Liouville BVPs as mostly beyond the scope of the discussion. In case you are curious, here are some general remarks concerning how these problems are most often solved.

The most common technique by far used to solve Sturm-Liouville BVPs is some variant of a power series solution. We have seen this technique already in the case of Bessel's equation - using that technique in section 11.3 we were able to solve

$$s^2 \frac{d^2 S(s)}{ds^2} + s \frac{dS(s)}{ds} + (s^2 - n^2)\, S(s) = 0$$

by assuming that

$$S(s) = s^{\alpha} \sum_{k=0}^{\infty} a_k\, s^k$$

and solving for α and the coefficients a_k:

$$\alpha^2 - n^2 = 0$$

$$S(s) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(n+j)!} \left(\frac{s}{2}\right)^{2j+n}$$

There are many other standard ODEs that can be solved by this approach. Further examples include the Legendre differential equation

$$\left(1 - x^2\right) \frac{d^2 y}{dx^2} - 2\, x \frac{d y}{dx} + l \left(l + 1\right) y = 0$$

Some of the solutions of the Legendre equation are Legendre polynomials.

and the Airy equation

$$\frac{d^2 y}{d x^2} - k^2\, x\, y = 0$$

whose solutions are the Airy functions.

Some Examples in Mathematica

The accompanying Mathematica notebook shows some examples of Sturm-Liouville BVPs and their solutions. As you will see in those examples, very often Mathematica will solve a problem by applying a preliminary change of variables or other transformation to the problem to convert the ODE to Bessel's equation, the Legendre equation, or Airy's equation. After that transformation solutions will then be expressed in terms of the standard functions that solve those equations.