Green's Function for the Heat Equation

We have now seen some examples of how Green's functions can be used to construct solutions to BVPs. The next step is to apply the Green's function technique to solve some PDE problems.

The first technical issue to deal with is the form of the solution. We saw when solving BVPs that solutions would take the form of integrals that looked like this:

$$u(x) = \int_a^b G(x;y)\, f(y)\, dy$$

The obvious generalization of this idea to functions of two variables looks like this:

$$u(x,t) = \int_0^\infty \int_a^b G(x,t;y,s)\, f(y,s)\, dy\, ds$$

For the Heat Equation, we are going to derive Green's function solutions by two methods: forcing a known solution into the form required for a Green's function solution, and using a delta function to derive the form of the Green's function.

Converting a known solution into the Green's function form

Consider the nonhomogeneous heat equation with Dirichlet boundary conditions on the interval $[0,l]$.

$$\frac{\partial u}{\partial t} - k\, \frac{\partial^2 u}{\partial x^2} = f(x,t)$$

$$u(x,0) = 0$$

$$u(0,t) = u(l,t) = 0$$

We have already solved this equation by the method of Fourier series. Solutions take the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi}{l} x\right)$$

where

$$a_n(t) = \int_0^t e^{-(k n^2 \pi^2 (t-s)/l^2)}\, c_n(s)\, ds$$

and $c_n(s)$ are the Fourier sine coefficients of the forcing function.

$$c_n(s) = \frac{2}{l} \int_0^l f(y,s) \sin\left(\frac{n\pi}{l} y\right) dy$$

Substituting all of these expressions into the expression for $u(x,t)$ allows us to recast the solution as a double integral.

$$u(x,t) = \sum_{n=1}^{\infty} \int_0^t e^{-(k n^2 \pi^2 (t-s)/l^2)}\, \frac{2}{l} \int_0^l f(y,s) \sin\left(\frac{n\pi}{l} y\right) dy\, ds\, \sin\left(\frac{n\pi}{l} x\right)$$

It is relatively easy to reorganize this expression to look like a double integral of a Green's function multiplied against $f(y,s)$:

$$u(x,t) = \int_0^t \int_0^l \frac{2}{l} \sum_{n=1}^{\infty} e^{-(k\, n^2\, \pi^2\, (t-s)/l^2)} \sin\left(\frac{n\,\pi}{l}\, y\right) \sin\left(\frac{n\,\pi}{l}\, x\right) f(y,s)\, dy\, ds$$

The only small problem remaining here is that the double integral is supposed to have its s integration range from 0 to ∞, not from 0 to t. This is easy to fix by modifying the definition of the Green's function to vanish for $s > t$:

$$G(x,t;y,s) = \begin{cases} \frac{2}{l} \sum_{n=1}^{\infty} e^{-(k\, n^2\, \pi^2\, (t-s)/l^2)} \sin\left(\frac{n\,\pi}{l}\, y\right) \sin\left(\frac{n\,\pi}{l}\, x\right) & 0 \le s \le t \\ 0 & s > t \end{cases}$$

Duhamel's Principle

Theorem If the function $v(t;r)$ solves the initial value problem

$$\frac{\mathrm{d}u_1(t)}{\mathrm{d}t} + a\, u_1(t) = 0$$

$$u_1(0) = f(r)$$

then the function

$$u_2(t) = \int_0^t v(t-r;r)\, dr$$

solves the initial value problem

$$\frac{\mathrm{d}u_2(t)}{\mathrm{d}t} + a\, u_2(t) = f(t)$$

$$u_2(0) = 0$$

Proof We verify by directly substituting the expression for $u_2(t)$ into the second differential equation:

$$\frac{\mathrm{d}u_2(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t v(t-r;r)\, dr\right)$$

$$= v(t-t;t) + \int_0^t \frac{\mathrm{d}}{\mathrm{d}t} v(t-r;r)\, dr$$

$$= v(0;t) + \int_0^t -a\, v(t-r;r)\, dr$$

$$= f(t) - a\ u_2(t)$$

Thus we see that

$$\frac{du_2(t)}{dt} + a\ u_2(t) = f(t)$$

Also, $u_2(t)$ satisfies the required initial condition:

$$u_2(0) = \int_0^0 v(-r;r)\ dr = 0$$

Observation The primary utility of Duhamel's principle is that it allows us to convert a problem in which $f(t)$ appears as a function into a problem in which $f(r)$ appears as a parameter. In solving the u_1 problem we can treat $f(r)$ as if it were a constant, making the problem much easier to solve than in the u_2 version of the problem. Of course, $f(t)$ eventually reasserts itself as a function when we go to construct the u_2 solution as an integral over u_1 solutions parameterized by r.

Generalization Duhamel's principle generalizes to higher order ODEs as follows.

Theorem Let $v(t;r)$ be the solution to the initial value problem

$$\frac{d^{(k)} u_1(t)}{dt^{(k)}} + a_{k-1} \frac{d^{(k-1)} u_1(t)}{dt^{(k-1)}} + a_{k-2} \frac{d^{(k-2)} u_1(t)}{dt^{(k-2)}} + \cdots + a_1 \frac{du_1(t)}{dt} + a_0\ u_1(t) = 0$$

$$u_1(0) = 0$$

$$\frac{du_1}{dt}(0) = 0$$

$$\vdots$$

$$\frac{d^{(k-2)} u_1}{dt^{(k-2)}}(0) = 0$$

$$\frac{d^{(k-1)} u_1}{dt^{(k-1)}}(0) = f(r)$$

then the function

$$u_2(t) = \int_0^t v(t-r;r)\ dr$$

solves the initial value problem

$$\frac{d^{(k)} u_2(t)}{dt^{(k)}} + a_{k-1} \frac{d^{(k-1)} u_2(t)}{dt^{(k-1)}} + a_{k-2} \frac{d^{(k-2)} u_2(t)}{dt^{(k-2)}} + \cdots + a_1 \frac{du_2(t)}{dt} + a_0\ u_2(t) = f(t)$$

$$u_2(0) = 0$$

$$\frac{\mathrm{d}u_2}{\mathrm{d}t}(0) = 0$$

$$\vdots$$

$$\frac{\mathrm{d}^{(k-2)}u_2}{\mathrm{d}t^{(k-2)}}(0) = 0$$

$$\frac{\mathrm{d}^{(k-1)}u_2}{\mathrm{d}t^{(k-1)}}(0) = 0$$

Alternative derivation of the Green's function for the heat equation

We have also seen that there is another way to characterize a Green's function. The Green's function is the response of the system to a forcing function that looks like a delta function. If we use this interpretation, we see that we can derive the Green's function for the heat equation by solving the PDE

$$\frac{\partial u}{\partial t} - k\,\frac{\partial^2 u}{\partial x^2} = \delta(x-y)\ \delta(t-s)$$

$$u(x,0) = 0$$

$$u(0,t) = u(l,t) = 0$$

Note that we need delta functions for both x and t here, since the idea is to apply an impulse that is isolated to single point in both space and time.

We can solve this PDE by the method of Fourier series. As usual, we start by multiplying both sides of the equation by Fourier sine functions and integrating in space:

$$\frac{2}{l}\int_0^l \left(\frac{\partial u}{\partial t} - k\,\frac{\partial^2 u}{\partial x^2}\right)\sin\left(\frac{n\,\pi\,x}{l}\right)dx = \frac{2}{l}\int_0^l \delta(x-y)\ \delta(t-s)\ \sin\left(\frac{n\,\pi\,x}{l}\right)dx$$

On the left we substitute the presumed form of the solution

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\ \sin\left(\frac{n\,\pi}{l}\,x\right)$$

and on the right we use the property of the delta function to get

$$\frac{\mathrm{d}a_n(t)}{\mathrm{d}t} + k\,\frac{n^2\,\pi^2}{l^2}\,a_n(t) = \frac{2}{l}\ \delta(t-s)\ \sin\left(\frac{n\,\pi\,y}{l}\right)$$

The initial conditions for the PDE also give us

$$a_n(0) = 0$$

This ODE for $a_n(t)$ can be solved by an application of Duhamel's principle. We convert the problem into a problem that takes the form

$$\frac{\mathrm{d}b_n(t)}{\mathrm{d}t} + k\,\frac{n^2\,\pi^2}{l^2}\,b_n(t) = 0$$

$$b_n(0) = \frac{2}{l}\,\delta(r\text{-}s)\,\sin\left(\frac{n\,\pi\,y}{l}\right)$$

In this problem we get to treat n, r, s, and y as parameters and effectively carry them along as constants. It is easy to see that this latter problem has solution

$$b_n(t;r) = \frac{2}{l}\,\delta(r\text{-}s)\,\sin\left(\frac{n\,\pi\,y}{l}\right)\,e^{-k\,n^2\,\pi^2\,t/l^2}$$

Duhamel's principle then tells us that the solution to

$$\frac{\mathrm{d}a_n(t)}{\mathrm{d}t} + k\,\frac{n^2\,\pi^2}{l^2}\,a_n(t) = \frac{2}{l}\,\delta(t\text{-}s)\,\sin\left(\frac{n\,\pi\,y}{l}\right)$$

$$a_n(0) = 0$$

is

$$a_n(t) = \int_0^t b_n(t\text{-}r;r)\,dr = \int_0^t \frac{2}{l}\,\delta(r\text{-}s)\,\sin\left(\frac{n\,\pi\,y}{l}\right)\,e^{-k\,n^2\,\pi^2(t\text{-}r)/l^2}\,dr$$

$$= \begin{cases} \frac{2}{l}\,\sin\left(\frac{n\,\pi\,y}{l}\right)\,e^{-k\,n^2\,\pi^2(t\text{-}s)/l^2} & t > s \\ 0 & t < s \end{cases}$$

Thus we see that the solution to the original problem

$$\frac{\partial u}{\partial t} - k\,\frac{\partial^2 u}{\partial x^2} = \delta(x\text{-}y)\,\delta(t\text{-}s)$$

$$u(x,0) = 0$$

$$u(0,t) = u(l,t) = 0$$

is

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\sin\left(\frac{n\,\pi\,x}{l}\right)$$

or

$$u(x,t) = \sum_{n=1}^{\infty} \left(\begin{cases} \frac{2}{l} \sin\left(\frac{n\pi y}{l}\right) e^{-k n^2 \pi^2 (t-s)/l^2} & t > s \\ 0 & t < s \end{cases} \right) \sin\left(\frac{n\pi}{l} x\right)$$

which leads to

$$G(x,t;y,s) = \begin{cases} \frac{2}{l} \sum_{n=1}^{\infty} e^{-k n^2 \pi^2 (t-s)/l^2} \sin\left(\frac{n\pi y}{l}\right) \sin\left(\frac{n\pi}{l} x\right) & 0 \le s \le t \\ 0 & s > t \end{cases}$$

This is the same expression for the Green's function we derived earlier.