The Wave Equation

In section 7.1 we use d'Alembert's method to solve the homogeneous wave equation.

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

$$u(0,t) = u(l,t) = 0$$

$$u(x,0) = \psi(x)$$

$$\frac{\partial u}{\partial t}(x,0) = \gamma(x)$$

The d'Alembert solution unfortunately can only solve the homogeneous form of the wave equation,

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

If we need to solve the inhomogeneous form

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = f(x,t)$$

we will have to develop an alternative technique. The next most obvious technique that suggests itself is the method of Fourier series.

Given the Dirichlet boundary conditions on the wave equation, we should expect solutions of the wave equation to take the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\left(\frac{n\pi}{l} x\right)$$

Applying the usual technique of multiplying both sides of the inhomogeneous wave equation by test functions $\sin(n\pi/l\, x)$ and integrating over the entire interval gives

$$\int_0^l \left(\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2}\right) \sin\left(\frac{n\pi}{l} x\right) \mathrm{d}x = \int_0^l f(x,t) \sin\left(\frac{n\pi}{l} x\right) \mathrm{d}x$$

or

$$\sum_{n=1}^{\infty} \left(\frac{\mathrm{d}^2 a_n(t)}{\mathrm{d}t^2} + \frac{c^2 n^2 \pi^2}{l^2} a_n(t)\right) \sin\left(\frac{n\pi}{l} x\right) = \sum_{n=1}^{\infty} c_n(t) \sin\left(\frac{n\pi}{l} x\right)$$

where the $c_n(t)$ are the Fourier sine coefficients of $f(x,t)$,

$$c_n(t) = \frac{2}{l}\int_0^l f(x,t)\,\sin\left(\frac{n\,\pi}{l}\,x\right)\mathrm{d}\,x$$

and the terms $a_n(t)$ are the Fourier sine coefficients of $u(x,t)$:

$$a_n(t) = \frac{2}{l}\int_0^l u(x,t)\,\sin\left(\frac{n\,\pi}{l}\,x\right)\mathrm{d}\,x$$

Thus we see that the problem boils down to solving ODEs of the form

$$\frac{\mathrm{d}^2 a_n(t)}{\mathrm{d}\,t^2} + \frac{c^2\,n^2\,\pi^2}{l^2}\,a_n(t) = c_n(t)$$

The initial conditions for these ODEs are provided by the initial conditions for the PDE above. The first condition gives

$$u(x,0) = \sum_{n=1}^{\infty} a_n(0)\,\sin\left(\frac{n\,\pi}{l}\,x\right) = \sum_{n=1}^{\infty} b_n\,\sin\left(\frac{n\,\pi}{l}\,x\right) = \psi(x)$$

or

$$a_n(0) = b_n$$

where the terms b_n are the Fourier sine coefficients of $\psi(x)$

$$b_n = \frac{2}{l}\int_0^l \psi(x)\,\sin\left(\frac{n\,\pi}{l}\,x\right)\mathrm{d}\,x$$

The second condition gives

$$\frac{\partial\,u}{\partial\,t}(x,0) = \sum_{n=1}^{\infty} a_n'(0)\,\sin\left(\frac{n\,\pi}{l}\,x\right) = \sum_{n=1}^{\infty} d_n\,\sin\left(\frac{n\,\pi}{l}\,x\right) = \gamma(x)$$

or

$$a_n'(0) = d_n$$

where the terms d_n are the Fourier sine coefficients of $\gamma(x)$

$$d_n = \frac{2}{l}\int_0^l \gamma(x)\,\sin\left(\frac{n\,\pi}{l}\,x\right)\mathrm{d}\,x$$

Putting this all together, we now have a set of ODEs with initial conditions:

$$\frac{\mathrm{d}^2 a_n(t)}{\mathrm{d}\,t^2} + \frac{c^2\,n^2\,\pi^2}{l^2}\,a_n(t) = c_n(t)$$

$$a_n(0) = b_n$$

$$a_n'(0) = d_n$$

These are second order, constant coefficient nonhomogeneous ODEs that can be solved directly by using techniques from Math 210 or Mathematica's DSolve function. The accompanying Mathematica notebook will show some examples.